2015 May ODE/PDE Preliminary Examination

Part I: ODE. Do 3 of the following 4 problems. You must clearly indicate which 3 are to be graded. Problems 1, 2, and 3 will be graded if no indication is given. Strive for clear and detailed solutions.

1. Let A be an $n \times n$ time-independent matrix.

   a) Prove that e^{At} is a fundamental matrix for the system $\dot{x} = Ax$.

   b) Prove for any continuous function $f(t)$, $x(t) = e^{At}(C + \int_0^t e^{-A\tau} f(\tau)\, d\tau)$ is the general solution for the system $\dot{x} = Ax + f(t)$.

   c) Let $A = \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & -2 & -1 \end{bmatrix}$. Find e^{At}.

2. Let $f(t,x) \in C^1(\mathbb{R}^{n+1})$. Prove that for all $t \geq 0$, if there is a function $k(t)$ such that $\|f(t,x)\| \leq k(t)\|x\|$ for all $x \in \mathbb{R}^n$, then the solution $x(t)$ of the initial value problem $\dot{x} = f(t,x)$, $x(0) = x_0$ satisfies

$$||x(t)|| \leq \|x_0\| e^{\int_0^t k(s)ds}.$$

3. Investigate the stability of the origin of the system

$$\begin{aligned} \dot{x_1} &= -x_2 + x_1 x_3 \\ \dot{x_2} &= x_1 + x_2 x_3 \\ \dot{x_3} &= -x_3 - x_1^2 - x_2^2 + x_3^2 \end{aligned}$$

4. Prove that the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2(1 - 3x_1^2 - 2x_2^2) \end{aligned}$$

   has a periodic orbit.

MAY 2015. **PRELIMINARY EXAMINATION**

Partial Differential Equations

Do three out of four problems below. Clearly indicate in the following boxes which three problems to be graded; otherwise problems 1, 2, and 3 will be used for grading:

1. Let U be a semi-infinite strip in $\mathbb{R}^2$ defined by

$$U = \{(x, y) \in \mathbb{R}^2 : 0 < y < L, 0 < x < \infty\}.$$

Let $u(x, y) \in C^2(U) \cap C(\bar{U})$ be a classical solution of the following problem

$$u_{xx} + u_{yy} = 0 \quad \text{in } U,$$

$$u(x, 0) = u(x, L) = 0 \quad \text{for all } x \in (0, \infty),$$

$$u(0, y) = 0 \quad \text{for all } y \in [0, L].$$

Assume that there are positive numbers N and C such that

$$|u(x, y)| \leq C + x^N \quad \text{for all } (x, y) \in U.$$

Prove that $u(x, y) = 0$ for all $(x, y) \in U$.

2. Let $D = U \times (0, T]$, where U is a bounded domain in $\mathbb{R}^n$ and $T > 0$.

(a) State without proof the maximum principle for the classical solution of the heat equation $v_t - \Delta v = 0$ in D.

(b) Prove that there is at most one classical solution $u \in C^{2,1}_{x,t}(D) \cap C(\bar{D})$ of the following initial boundary value problem:

$$u_t - \Delta u = |\nabla u|^2 \quad \text{in } D,$$

$$u(x, 0) = h(x) \quad \text{on } U,$$

$$u(x, t) = g(x, t) \quad \text{on } \partial U \times (0, T],$$

where $h(x)$ and $g(x, t)$ are given initial and boundary data.

(*Hint: You can try to use $v = \Phi(u)$ with an appropriate function Φ.*)

3. Assume $u(x,t)$ is the unique classical solution of the following Cauchy problem

$$u_t = -xu_x \quad \text{in } \mathbb{R} \times (0,\infty),$$

$$u(x,0) = \phi(x) \quad \text{for all } x \in \mathbb{R},$$

where $\phi(x) \in C^1(\mathbb{R})$ is a given function that satisfies $\phi(0) = 0$.

Prove for any $x \in \mathbb{R}$ that

$$\lim_{t\to\infty} u(x,t) = 0.$$

4. Let U be a bounded domain in $\mathbb{R}^n$. Suppose $u(x,t) \in C^2(\bar{U} \times [0,\infty))$ is a classical solution of the problem

$$\frac{\partial^2 u}{\partial t^2} = \Delta u - \frac{\partial u}{\partial t} \quad \text{in } U \times (0,\infty),$$

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = 0 \quad \text{on } U,$$

$$u(x,t) = h(x,t) \quad \text{on } \partial U \times [0,\infty),$$

where $h(x,t)$ is a given continuous function on $\partial U \times [0,\infty)$.

Assume there is a function $H(x,t) \in C^2(\bar{U} \times [0,\infty))$ such that $H(x,t) = h(x,t)$ on $\partial U \times [0,\infty)$, and $H(x,0) = H_t(x,0) = 0$ on U.

Prove that there exists a positive constant C such that for any $t > 0$ one has

$$\int_0^t \int_U |u_t(x,\tau)|^2 dx d\tau \le C\Big(\int_0^t \int_U |H_t(x,\tau)|^2 dx d\tau + \int_0^t e^{-(t-\tau)} G(\tau) d\tau\Big),$$

where

$$G(t) = \int_0^t \int_U F^2(x,\tau) dx d\tau$$

with $F(x,t) = \frac{\partial^2 H}{\partial t^2} - \Delta H + \frac{\partial H}{\partial t}$.