

AUGUST 2015. **PRELIMINARY EXAMINATION**
Partial Differential Equations

Do three out of four problems below. Clearly indicate in the following boxes which three problems to be graded; otherwise problems 1, 2, and 3 will be used for grading:

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1. Let D be a bounded domain in \mathbb{R}^n with C^1 -boundary, and $U = D \times (0, \infty)$. Let Γ denote the parabolic boundary of U .

Let α_1, α_2 be two positive numbers.

Let $u_1(x, t)$ and $u_2(x, t)$ with $(x, t) \in \bar{U}$ be two functions in $C^2(\bar{U})$.

Suppose

$$\frac{\partial u_1}{\partial t} - \alpha_1 \Delta u_1 = \frac{\partial u_2}{\partial t} - \alpha_2 \Delta u_2 \quad \text{on } U,$$

and

$$u_1 = u_2 \quad \text{on } \Gamma.$$

Assume there is $M > 0$ such that

$$\int_D \left(|\nabla u_1(x, t)|^2 + |\nabla u_2(x, t)|^2 \right) dx \leq M \quad \forall t > 0.$$

Prove that there exists a constant $C > 0$ such that

$$\int_D |u_1(x, t) - u_2(x, t)|^2 dx \leq C |\alpha_1 - \alpha_2|^2 \quad \forall t \geq 0.$$

(Note: Poincaré's inequality can be used without proof.)

2. Let D be a bounded domain in \mathbb{R}^n and $U = D \times (0, \infty)$. Let Γ denote the parabolic boundary of U . Suppose $u(x, t)$ is a classical solution of the problem

$$u_t(x, t) - \Delta u(x, t) = 1 \quad \forall (x, t) \in U,$$

and

$$u(x, t) = 1 \quad \forall (x, t) \in \Gamma.$$

Prove that for any $\alpha > 1$ and $x \in D$ one has

$$\lim_{t \rightarrow \infty} \left(\frac{u(x, t)}{t^\alpha} \right) = 0.$$

3. Let $u(x, t)$ be the classical solution of the wave equation

$$u_{tt}(x, t) - u_{xx}(x, t) = 0 \quad \text{in } \mathbb{R} \times (0, \infty)$$

satisfying the initial conditions

$$u(x, 0) = g(x) \quad \text{and} \quad u_t(x, 0) = h(x),$$

where $g(x)$ and $h(x)$ are given functions on \mathbb{R} .

Assume

$$\lim_{x \rightarrow \pm\infty} g(x) = 0,$$

$$h(x) \geq 0 \quad \forall x \in \mathbb{R} \quad \text{and} \quad \int_{-\infty}^{+\infty} h(x) dx < \infty.$$

Prove for any fixed $t > 0$ that

$$\lim_{x \rightarrow \pm\infty} u(x, t) = 0.$$

4. Suppose $u(x)$ is a classical, bounded solution of the Laplace equation in \mathbb{R}^n , that is, $u \in C^2(\mathbb{R}^n)$, $|u(x)| \leq C$ and $\Delta u(x) = 0$ for all $x \in \mathbb{R}^n$, where C is a positive constant.

Prove that $u = \text{constant}$ on \mathbb{R}^n .