MAY 2016 ODE PRELIMINARY EXAM

You must clearly indicate which 3 problems are to be graded by circling the problem numbers on the exam. If you fail to clearly identify 3 problems, then problems 1, 2, and 3 will be graded.

1. Consider the ordinary differential equation (you may answer/solve the sub-problems below in any order) :

$$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} = \frac{1}{2\pi}\ln(r); \quad y(1), y'(1) \text{ given.}$$

   (a) Find the fundamental matrix $\Phi(r, 1)$ for the system where $r \in (0, \infty)$.

   (b) Find the state transition matrix $\Phi(r, x)$ for the system where $r, x \in (0, \infty)$.

   (c) Find a formula for y as a function of r on $(0, \infty)$.

   Hint: You may use the facts: $\int x\ln(x)\,dx = \frac{x^2}{4}(2\ln(x) - 1) + C$, and $\int x\ln^2(x)\,dx = \frac{x^2}{4}(2\ln^2(x) - 2\ln(x) + 1) + C$. These may or may not be needed depending on the method used.

2. Consider the system:

$$\dot{x} = x(1 - x^2 - y^2) - y^2$$
$$\dot{y} = y(1 - x^2 - y^2) + x\,y$$

   (a) Find all equilibrium points for the system.

   (b) Determine the nature of the equilibrium points.

   (c) Does the system admit any periodic orbits?

3. (a) Consider the system:

$$\begin{aligned}\dot{x}_1 &= -\frac{1}{T}x_1 + \tanh(\lambda\,x_1) - \tanh(\lambda\,x_2)\\ \dot{x}_2 &= -\frac{1}{T}x_2 + \tanh(\lambda\,x_1) + \tanh(\lambda\,x_2),\end{aligned}$$

   where λ and T are positive constants. Using the fact that $|\tanh(u)| < 1$ for all real u, show that the function $r = \sqrt{x_1^2 + x_2^2}$ satisfies the differential inequality

$$\dot{r} \le -\frac{1}{T}r + 2\sqrt{2}.$$

   (b) Show that the solution $x(t) = (x_1(t), x_2(t))$ satisfies the inequality

$$\|x(t)\|_2 \le e^{-\frac{t}{T}}\|x(0)\|_2 + 2\sqrt{2}\,T\,(1 - e^{-\frac{t}{T}}).$$

4. Consider the system:

$$\dot{x}_1 = -x_2 + x_1 x_3$$
$$\dot{x}_2 = x_1 + x_2 x_3$$
$$\dot{x}_3 = -x_3 - x_1^2 - x_2^2 + x_3^3$$

(a) Find an expression for the center manifold of the system near the origin.

(b) Determine the dynamics of the system on the center manifold in polar coordinates.

(c) Determine the stability of the origin.

MAY 2016. **PRELIMINARY EXAMINATION**

Partial Differential Equations

Do three out of four problems below. Write in the following boxes the three problems that have to be graded, otherwise problems 1, 2, and 3 will be used for grading:

1. Let $n \geq 3$ and denote $B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$ for $r > 0$.

Let $U = B(0,1)$ and $u(x) \in C^2(\bar{U} \setminus \{0\})$ be a classical solution of the problem

$$\Delta u = 0 \quad \text{in } U \setminus \{0\},$$

$$u = 0 \quad \text{on } \partial U.$$

For $r \in (0,1]$, let

$$M(r) = \sup_{x \in \partial B(0,r)} |u(x)|.$$

Assume

$$\lim_{r \to 0} r^{n-2} M(r) = 0.$$

Prove that $u(x) = 0$ for all $x \in \bar{U} \setminus \{0\}$.

2. Let U be a bounded domain in $\mathbb{R}^n$. Assume $u(x,t) \in C^{2,1}_{x,t}(\bar{U} \times [0,\infty))$ is a classical solution of the problem

$$u_t - \Delta u = f(x) \quad \text{in } U \times (0,\infty),$$

$$u(x,t) = 0 \quad \text{on } \partial U \times (0,\infty).$$

$$u(x,0) = g(x) \quad \text{on } U,$$

where $f(x)$ and $g(x)$ are given functions.

(a) Prove for all $t > 0$ that

$$\int_U u^2(x,t)dx \leq e^{-c_1 t} \int_U g^2(x)dx + c_2 \int_U f^2(x)dx,$$

where c_1 and c_2 are positive constants independent of u, f, g, t.

(b) Prove for all $t > 0$ that

$$\int_U |\nabla u(x,t)|^2 dx \le e^{-c_3 t} \int_U \left(g^2(x) + |\nabla g(x)|^2\right) dx + c_4 \int_U f^2(x) dx,$$

where c_3 and c_4 are positive constants independent of u, f, g, t.

(Note: You can use Poincaré's inequality without proof.)

3. Denote $U = \mathbb{R} \times (0, +\infty)$. Let $u = u(x,t) \in C^2(\bar{U})$ be the classical solution of the problem

$$u_{tt} = u_{xx} \quad \text{in } U,$$

$$u(x,0) = g(x) \quad \text{and} \quad u_t(x,0) = h(x) \quad \text{in } \mathbb{R}.$$

Assume that $g(x)$ and $h(x)$ are analytic functions on $\mathbb{R}$.

Prove that $u(x,t)$ is analytic in two variables x and t on U, that is, for any $(x_0, t_0) \in U$, there are numbers $c_{i,j}$ for integers $i, j \ge 0$, and $\delta > 0$ such that the series

$$\sum_{n=0}^{\infty} A_n(x,t), \quad \text{where } A_n(x,t) = \sum_{0 \le i \le n} c_{i,n-i}(x - x_0)^i (t - t_0)^{n-i},$$

converges uniformly on $\{(x,t) \in U : |x - x_0| < \delta, |t - t_0| < \delta\}$.

4. Prove that there is an entropy solution $u(x,t)$ of the following problem

$$u_t + 8uu_x = 0 \quad \text{on } \mathbb{R} \times (0, \infty),$$

$$u(x,0) = g(x) \quad \text{on } \mathbb{R},$$

where

$$g(x) = \begin{cases} 2, & \text{if } x < 0, \\ 0, & \text{if } 0 < x < 2, \\ 1, & \text{if } 2 < x, \end{cases}$$

such that

(a) for each $x \in \mathbb{R}$

$$\lim_{t \to \infty} u(x,t) = 2,$$

and

(b) for each $t > 0$

$$\lim_{x \to \infty} u(x,t) = 1.$$