AUG 2016 ODE PRELIMINARY EXAM

You must clearly indicate which 3 problems are to be graded by circling the problem numbers on the exam. If you fail to clearly identify 3 problems, then problems 1, 2, and 3 will be graded.

1. Does there exist x_0 so that the system $\dot{x}(t) = A(t)\, x(t)$, $x(0) = x_0$ has a periodic solution with non-zero period, where:

$$A(t) = \begin{bmatrix} \cos(t) & -1 \\ 1 & \cos(t) \end{bmatrix}?$$

2. Consider the two dimensional system:

$$\dot{x}_1 = x_2 \cos(x_1); \quad \dot{x}_2 = \sin(x_1). \tag{1}$$

(a) Find all equilibrium points and classify their types.
(b) Prove that this system does not have a periodic solution.
(c) Prove that there is no compact set in $\mathbb{R}^2$ that is positively invariant (that is, trajectories starting in the set for some time t_0 stay in the set for all time $t \geq t_0$).

3. Let $f(x,t)$ be continuous in t and continuously differentiable in x on $E \times [t_0, t_1]$, where $E \subset \mathbb{R}^n$ is an open set. Let K be a compact subset of E and $x_0 \in K$. Let $x(t)$ be the solution of $\dot{x}(t) = f(x,t)$; $x(t_0) = x_0$. Suppose that $x(t)$ is defined and $x(t) \in K$ for all $t \in [t_0, T)$ where $T < t_1$.
   (a) Show that $x(t)$ is uniformly continuous on $[t_0, T)$.
   (b) Show that $x(T)$ is defined and belongs to W and $x(t)$ is a solution on $[t_0, T]$.
   (c) Show that there is $\delta > 0$ such that the solution can be extended to $[t_0, T + \delta]$.

4. Consider the system:

$$\begin{aligned} \frac{dx_1}{dt} &= f(x_3) - b_1 x_1 \\ \frac{dx_2}{dt} &= a_1 x_1 - b_2 x_2 \\ \frac{dx_3}{dt} &= a_2 x_2 - b_3 x_3, \end{aligned}$$

where a_1, a_2, b_1, b_2, b_3 are positive real numbers, and f is a positive and monotone decreasing function. Assume f to be continuously differentiable.

   (a) Show that there exists an equilibrium point $\mathrm{x}_0 = (x_{01}, x_{02}, x_{03})$ in the first octant (that is, $x_{0i} > 0$ for $i = 1, 2, 3$).

   (b) By the Routh-Hurwitz criterion, all the roots of a third order polynomial $P(s) = a_3\, s^2 + a_2\, s^2 + a_1\, s + a_0$ have negative real parts if and only if all coefficients are positive and $a_2\, a_1 > a_3\, a_0$. Find a condition on $f'(x_{03})$ that will guarantee that the equilibrium point in the first octant for the above system is asympotically stable.

AUGUST 2016. PRELIMINARY EXAMINATION

Partial Differential Equations

Do three out of four problems below. Write in the following boxes the three problems that have to be graded, otherwise problems 1, 2, and 3 will be used for grading:

1. Let $U = \{(x, y) \in \mathbb{R}^2 : 0 < x < \infty, 0 < y < x\}$ and $u(x, y) \in C^2(U) \cap C(\bar{U})$ be a classical solution of the problem

$$\Delta u = 0 \quad \text{in } U,$$

$$u = 0 \quad \text{on } \partial U.$$

For $x \in (0, \infty)$, let

$$M(x) = \sup_{0 \le y \le x} |u(x, y)|.$$

Assume

$$\lim_{x \to \infty} \frac{M(x)}{x} = 0.$$

Prove that $u(x, y) = 0$ for all $(x, y) \in U$.

Hint: you can make use of harmonic functions $(x + a)^2 - y^2$ *for any* $a > 0$.

2. Let U be a bounded domain in $\mathbb{R}^n$ with C^1-boundary. Assume $u(x, t) \in C^{2,1}_{x,t}(\bar{U} \times [0, \infty))$ is a classical solution of the problem

$$u_t(x, t) - (|x|^2 + 1)\Delta u(x, t) + 2u(x, t) = f(x) \quad \text{in } U \times (0, \infty),$$

$$u(x, t) = 0 \quad \text{on } \partial U \times (0, \infty).$$

$$u(x, 0) = g(x) \quad \text{in } U,$$

where $f(x)$ and $g(x)$ are given functions.

Prove for all $t > 0$ that

$$\int_U u^2(x, t)dx \le e^{-c_1 t} \int_U g^2(x)dx + c_2 \int_U f^2(x)dx,$$

where c_1 and c_2 are positive constants independent of u, f, g, t.

(Note: You can use Poincaré's inequality without proof.)

3. Let U be a bounded domain in $\mathbb{R}^n$ with C^1-boundary. Let $u = u(x,t) \in C^2(\bar{U} \times [0,\infty))$ be the classical solution of the problem

$$u_{tt} - \Delta u + \alpha u_t = f(x) \quad \text{in } U \times (0,\infty),$$

$$u(x,0) = g(x) \quad \text{and} \quad u_t(x,0) = h(x) \quad \text{in } U,$$

$$u(x,t) = 0 \quad \text{on} \quad \partial U \times (0,\infty),$$

where α is a positive constant, and $f(x)$, $g(x)$ and $h(x)$ are given functions.

Prove that there is $C > 0$ independent of u, f, h, g such that

$$\limsup_{t\to\infty} \left(\frac{1}{t} \int_U (|u_t(x,t)|^2 + |\nabla u(x,t)|^2) dx \right) \le C \int_U f^2(x) dx.$$

4. Solve for solution $u(x,y)$ of the following problem

$$u_x + u\, u_y = 1,$$

$$u(x,x) = x/2.$$