2017 May ODE/PDE Preliminary Examination

Part I: ODE. Do 3 of the following 4 problems. You must clearly indicate which 3 are to be graded. Problems 1, 2, and 3 will be graded if no indication is given. Strive for clear and detailed solutions.

1. Let A be an $n \times n$ matrix with all eigenvalues located in the left half complex plane $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$, and let $f(t)$ be any continuous function bounded on $[0, \infty)$. Prove that the solution $x(t)$ of the initial value problem

$$\dot{x} = Ax + f(t), \quad x(0) = x_0$$

is bounded on $[0, \infty)$ for any initial value x_0.

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function with the Lipschitz constant L. Prove that the solution $x(t)$ of the initial value problem

$$\dot{x} = f(x), \qquad x(0) = x_0$$

satisfies

$$\|x(t) - x_0\| \leq \frac{\|f(x_0)\|}{L}(e^{L|t|} - 1)$$

for all $t \in (-\infty, \infty)$.

3. Use Lyapunov function of the form $V(x) = ax_1^4 + bx_2^2 + cx_3^2$ to determine the stability of the system

$$\begin{aligned} \dot{x_1} &= -x_2 - x_3 + x_1^5 \\ \dot{x_2} &= x_3 - x_1^3 \\ \dot{x_3} &= -x_2 - x_1^3 \end{aligned}$$

at the origin.

4. Prove that the system

$$\begin{aligned} \dot{x}_1 &= -ax_1 + bx_2^2 \\ \dot{x}_2 &= ax_2 - x_1x_2 \end{aligned}$$

does not have any periodic orbit for any real numbers a, b.

MAY 2017. **PRELIMINARY EXAMINATION**

Partial Differential Equations

Do three out of four problems below. Clearly indicate in the following boxes which three problems have to be graded, otherwise problems 1, 2, and 3 will be used for grading.

1. Let $B = \{x \in \mathbb{R}^3 : |x| < 1\}$ be the ball of radius 1 centered at the origin in $\mathbb{R}^3$.

Let $u(x) \in C^2(B \setminus \{0\})$ be a classical solution of the problem

$$\Delta u = \frac{2}{|x|^2} \quad \text{in } B \setminus \{0\},$$
$$u(x) = 0 \quad \text{on the boundary } \partial B.$$

Let

$$M(r) = \sup_{|x|=r} |u(x)|, \quad 0 < r \leq 1,$$

and assume that

$$\lim_{r \to 0} [M(r)r] = 0. \tag{1.1}$$

Prove that

$$u(x) = 2 \ln |x| \quad \text{in } B \setminus \{0\}.$$

Hint: You may want to reduce the problem to the Laplace equation in $B \setminus \{0\}$ *for the function* $v(x) = u(x) - 2 \ln |x|$, *and then utilize property* (1.1). *Also, pay attention to the fact that this problem is for* $\mathbb{R}^3$.

2. Let U_+ be the half strip $\{(x,t) : x > 0, 0 < t \leq T\}$. Let $u(x,t)$ be a classical solution of the following initial value problem (IVP)

$$u_t - x^2 u_{xx} - x u_x = 0 \quad \text{in } U_+, \tag{2.1}$$

$$u(x,0) = 0, \quad x > 0. \tag{2.2}$$

Assume

$$|u| \leq C < \infty \quad \text{in } U_+. \tag{2.3}$$

(a) Use the substitution $x = e^z$ (or $z = \ln x$), to show that the function $v(z,t) = u(e^z, t)$ is a solution of the IVP:

$$v_t(z,t) - v_{zz}(z,t) = 0 \quad \text{in } \mathbb{R} \times (0,T],$$

$$v(z,0) = 0, \quad z \in \mathbb{R}.$$

(b) State without proof the maximum principle for the classical solution in the class of exponentially growing function for heat equation:

$$\eta_t - \eta_{zz} = 0 \quad \text{in } \mathbb{R} \times (0,T], \quad \text{and} \quad \eta(z,0) = \eta_0(z) \text{ in } \mathbb{R}.$$

Here $\eta_0(z)$ is a continuous and bounded function.

(c) Use the above maximum principle to prove that, under assumption (2.3), the solution of the original IVP (2.1) and (2.2) is

$$u(x,t) \equiv 0.$$

3. Let $D = U \times (0,\infty)$, where U is a bounded domain in $\mathbb{R}^n$. Let $u = u(x,t) \in C^{2,1}_{x,t}(\bar{D})$ be a classical solution of the problem

$$a(t)u_t - \Delta u = 0 \quad \text{in } D,$$

$$u(x,0) = u_0(x) \quad \text{on } U,$$

$$u(x,t) = 0 \quad \text{on } \partial U \times [0,\infty).$$

Let C_p be a positive constant such that the following Poincaré's inequality holds

$$C_p \int_U v^2 dx \leq \int_U |\nabla v|^2 dx, \tag{3.1}$$

for any $v \in C^1(\bar{U})$ which vanishes on the boundary ∂U.

Assume

(i) The function $a(t)$ belongs to $C^1([0,\infty))$ and $a(t) > 0$ for all $t \geq 0$. *Note that under this assumption we have* $0 < C_1(T) \leq a(t) \leq C_2(T) < \infty$ *for* $t \in [0,T]$*, where* $C_1(T)$ *and* $C_2(T)$ *are constants depending on* $T \in (0,\infty)$.

(ii) There exists $T_0 > 0$ such that the derivative $a'(t)$ satisfies

$$|a'(t)| \leq C_p, \quad \text{for all } t \geq T_0.$$

Here C_p is the constant inequality (3.1).

Let
$$I(t) = \int_U a(t)\, u^2(x,t)\, dx.$$

Prove that:

(a) there is a constant $C > 0$ such that
$$I(t) \le C \quad \text{for all } t \ge 0;$$

(b) if, additionally,
$$\int_0^\infty \frac{dt}{a(t)} = \infty,$$
then
$$\lim_{t\to\infty} I(t) = 0.$$

4. Let $u(x,t)$ be a classical solution of the Cauchy problem on the half line:
$$u_{tt} - u_{xx} = 0, \quad x, t > 0,$$
with the boundary condition
$$u_x(0,t) = 0, \quad t > 0,$$
and initial data
$$u(x,0) = x^2 \quad \text{and} \quad u_t(x,0) = 1, \quad x > 0.$$
Find the limit
$$\lim_{t\to\infty} \frac{u(x,t)}{t^2}, \quad \text{for } x > 0.$$

Hint: Use the method of reflection and D'Alambert's formula.