2017 August ODE/PDE Preliminary Examination

Part I: ODE. Do 3 of the following 4 problems. You must clearly indicate which 3 are to be graded. Problems 1, 2, and 3 will be graded if no indication is given. Strive for clear and detailed solutions.

1. Let A be a 2×2 real matrix with purely imaginary eigenvalues. Show that there exists a bounded function $f(t)$ such that the solution of the initial value problem

$$\dot{x} = Ax + f(t), \quad x(0) = x_0$$

is unbounded on $[0, \infty)$.

2. Let $E \subset \mathbb{R}^n$ be open and $f : E \mapsto \mathbb{R}^n$ be continuously differentiable and Lipschitz on E. Prove that the solution $\phi(t, y)$ of the initial value problem

$$\dot{x}(t) = f(x(t)), \qquad x(0) = y$$

is continuous with respect to y uniformly for all t in a closed and bounded interval I contained in all the intervals of existence of $\phi(t, z)$ for z in a neighborhood of y contained in E.

3. Determine the stability of the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 - x_2^2 - x_3^2 \\ \dot{x}_2 &= -x_3 + x_1 x_2 \\ \dot{x}_3 &= x_2 + x_1 x_3 \end{aligned}$$

at the origin.

4. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 - x_1^3 \\ \dot{x}_2 &= x_1 + x_2 - x_1^2 x_2 - x_2^3. \end{aligned}$$

Accept the fact without proof that the origin is the only equilibrium point of the system. Prove that the system has an asymptotically stable periodic orbit in

$$S = \{x : 1 \leq x_1^2 + x_2^2 \leq 2\}.$$

August 2017. PRELIMINARY EXAMINATION

Partial Differential Equations

Do three out of four problems below. Clearly indicate in the following boxes which three problems have to be graded, otherwise problems 1, 2, and 3 will be used for grading.

1. Let $u(x,t)$, for $(x,t) \in [0,\infty) \times [0,\infty)$, be a classical solution of the following problem on the half-line:

$$u_{tt} - u_{xx} = 0 \text{ in } \ U_+ = (0,\infty) \times (0,\infty),$$

with the boundary condition

$$u(0,t) = 1 \text{ for } \ 0 < t < \infty,$$

and the initial data

$$u(x,0) = x^2 \ \text{ and } \ u_t(x,0) = 1, \ \text{ for } x \in (0,\infty).$$

Find the value of

$$\lim_{t\to\infty} \frac{u(1,t)}{t^2}.$$

(Recommendation: You may, first reduce problem to the problem with homogeneous boundary condition, and then use method of reflection and d'Alembert's formula.)

2. Let $B = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\}$, and $u \in C^2(B) \cap C(\bar{B})$ be the classical solution of the problem

$$\Delta u = 0 \quad \text{in } B,$$

$$u(x_1, x_2) = x_1^2 \sin x_2 \quad \text{on the boundary } \partial B.$$

Prove that

$$u(0,0) = 0.$$

3. Denote $U_+ = (0,\infty) \times (0,T]$. Let $u(x,t)$ be a classical solution of the following initial, boundary value problem (IBVP):

$$u_t - u_{xx} = 0 \quad \text{in } U_+,$$

$$u(x,0) = 0, \quad \text{for all } 0 \le x < \infty$$

$$u(0,t) = 0, \quad \text{for all } 0 < t < \infty.$$

Assume that

$$|u(x,t)| \le e^{x^2} \quad \text{for all } (x,t) \in U_+. \tag{1}$$

(a) Using the method of reflection to reduce the original problem on U_+ to an initial value problem on

$$U = \mathbb{R} \times (0,T].$$

(b) State without proof the maximum principle for the classical solution in the class of exponentially growing functions for the heat equation on $\mathbb{R}$.

(c) Use the maximum principle in part (b) to prove that under assumption (1), the solution of the original IBVP on U_+ is

$$u(x,t) \equiv 0.$$

4. Let $D = U \times [t_0, \infty)$, where U is a bounded domain in $\mathbb{R}^n$, and t_0 is a fixed number in $\mathbb{R}$. Let $u = u(x,t) \in C^{2,1}_{x,t}(\bar{D})$ be a classical solution of the problem

$$u_t - \Delta u = \lambda u \quad \text{in } D,$$

$$u(x,t_0) = u_0(x) \text{ on } U,$$

$$u(x,t) = 0 \quad \text{on } \partial U \times [t_0, \infty).$$

Assume that

$$|\lambda| \le C_p,$$

where C_p is a positive constant for which the following Poincaré's inequality

$$C_p \int_U v^2 dx \le \int_U |\nabla v|^2 dx \tag{2}$$

holds for all $v \in C^1(U)$ which vanishes on the boundary ∂U.

Let

$$I(t) = \int_U u(x,t)^2 dx.$$

Prove that

$$I(t) \le \int_D u_0^2(x) dx$$

for all $t > t_0$.