May 2018 ODE/PDE Preliminary Examination

Part I: ODE. Do 3 of the following 4 problems. You must clearly indicate which 3 are to be graded. Problems 1, 2, and 3 will be graded if no indication is given. Strive for clear and detailed solutions.

1. Let $f(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and suppose there are $k_1,\ k_2 > 0$ such that
$$\|f(t,x)\| \leq k_1 + k_2\|x\|$$
for any $x \in \mathbb{R}^n$ and for all $t \in [t_0, \infty)$. Prove that the solution of
$$\dot{x} = f(t,x), \quad x(t_0) = x_0$$
satisfies
$$\|x(t)\| \leq \|x_0\| e^{k_2(t-t_0)} + \frac{k_1}{k_2}\left[e^{k_2(t-t_0)} - 1\right]$$
for all $t \geq t_0$ for which the solution exists.

2. Consider the linear system $\dot{x} = A(t)x$ where the square matrix function A is a continuous, periodic function with the smallest positive period ω. Such systems are called Floquet systems.
   Let $\Phi(t)$ be the fundamental matrix for the system. That is, $\Phi(t)$ is a square matrix function that satisfies the matrix equation $\dot{X} = A(t)X$ and $\det \Phi(t) \neq 0$ for all t. Then there is a square constant matrix M such that $\Phi(t+\omega) = \Phi(t)M$ for all t and the eigenvalues μ of $C := \Phi^{-1}(0)\Phi(\omega)$ are called the Floquet multipliers of the Floquet system.

   a. If $\mu = -1$ is a Floquet multiplier, prove there is a nontrival solution with period 2ω.

   b. Find the Floquet multipliers for the Floquet system
$$\dot{x} = \begin{bmatrix} 1-\cos t & 0 \\ \cos t & -1 \end{bmatrix} x.$$

3. Determine the stability of all equilibrium points of the system
$$\begin{aligned} \dot{x} &= -x - y^2 \\ \dot{y} &= -\frac{1}{2}y + 2xy. \end{aligned}$$
Identify a family of positively invariant sets.

4. This question has two parts.

   a. State the Poincare-Bendixson Theorem.

   b. Use it to show that the system
$$\begin{aligned} \dot{x} &= x - y + x^3 \\ \dot{y} &= x + y - y^3 \end{aligned}$$
has a periodic solution. Here the only equilibrium point is $(0,0)$.

MAY 2018. **PRELIMINARY EXAMINATION**

Partial Differential Equations

Do three out of four problems below. Clearly indicate in the following boxes which three problems have to be graded, otherwise problems 1, 2, and 3 will be used for grading.

1. Let Ω be the rectangular domain $\big\{(x,y) : a < x < b; c < y < d\big\}$.

Let $u(x,y) \in C^4(\bar{\Omega})$ be a classical solution of the problem

$$u_{xxxx} + u_{yyyy} = -u^{2n+1} \quad \text{in } \Omega, \quad n = 1, 2, 3, ...$$
$$u(x) = f(x,y) \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g(x,y) \quad \text{on the boundary } \partial\Omega,$$

where $\nu = \nu(x,y)$ is the outward normal vector to the boundary $\partial\Omega$.

Prove that the above solution $u(x,y)$ is unique.

Hint: You may use Poincaré's inequality for the functions $w(x,y)$, $w_x(x,y)$ and $w_y(x,y)$ such that $w = w_x = 0$ when $x = a$ and $x = b$, and $w = w_y = 0$ when $y = c$ and $y = d$.

2. Let $u(x) \in C^2(\mathbb{R}^n)$ be a classical solution of the equation $\Delta u = 0$ in $\mathbb{R}^n$, for $n \geq 2$, such that derivatives $u_{x_i} \geq C > -\infty$ for $i = 1, 2, \ldots, n$.

Prove that $u(x) = B + \sum_{i=1}^{n} b_i x_i$ for some constants $B, b_1, \ldots, b_n$.

Hint: You may use Liouville Theorem for harmonic functions.

3. Let $D = U \times (0, \infty)$, where U is a bounded domain in $\mathbb{R}^n$ with $n \geq 1$.

Let $u = u(x,t) \in C^{2,1}_{x,t}(\bar{D})$ be a classical solution of the following initial, boundary value problem for sine-Gordon-type equation

$$u_t - \beta \Delta u = \alpha \sin u \quad \text{in } D,$$

$$u(x,0) = u_0(x) \quad \text{on } U,$$

$$u(x,t) = f(x,t) \quad \text{on } \partial U \times [0,\infty),$$

where $\alpha \in \mathbb{R}$, and $\beta > 0$ are constants.

Prove that this solution $u(x,t)$ is unique.

Hint: You may use energy method .

4. Let $u(x,t)$ be a classical solution of the Cauchy problem for the wave equation:

$$u_{tt} - \Delta u = B(u_t, |\nabla u|) \quad x \in \mathbb{R}^n, \ t > 0,$$

with initial data

$$u(x,0) = A = \textit{constant} \quad \text{and} \quad u_t(x,0) = 0, \quad \text{for all} \quad |x| \le R,$$

where B is a function from $\mathbb{R}^2$ to $\mathbb{R}$, and the positive number R is fixed.

Assume that there exists a constant $C > 0$ such that

$$|B(z,\xi)| \le C(|z| + |\xi|) \quad \forall (z,\xi) \in \mathbb{R}^2.$$

Prove that

$$u(x,t) = A,$$

for all (x,t) such that $|x| < R - t$ and $0 \le t \le R$.

Hint: You may use the energy $E(t) = \int_{\{|x|<R-t\}} u_t^2(x,t) + |\nabla u(x,t)|^2 dx$.