May 2018 ODE/PDE Preliminary Examination

Part I: ODE. Do 3 of the following 4 problems. You must clearly indicate which 3 are to be graded. Problems 1, 2, and 3 will be graded if no indication is given. Strive for clear and detailed solutions.

1. Let \( f(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) be continuous and suppose there are \( k_1, k_2 > 0 \) such that
   \[
   \|f(t, x)\| \leq k_1 + k_2\|x\|
   \]
   for any \( x \in \mathbb{R}^n \) and for all \( t \in [t_0, \infty) \). Prove that the solution of
   \[
   \dot{x} = f(t, x), \quad x(t_0) = x_0
   \]
   satisfies
   \[
   \|x(t)\| \leq \|x_0\|e^{k_2(t-t_0)} + \frac{k_1}{k_2} [e^{k_2(t-t_0)} - 1]
   \]
   for all \( t \geq t_0 \) for which the solution exists.

2. Consider the linear system \( \dot{x} = A(t)x \) where the square matrix function \( A \) is a continuous, periodic function with the smallest positive period \( \omega \). Such systems are called Floquet systems.
   Let \( \Phi(t) \) be the fundamental matrix for the system. That is, \( \Phi(t) \) is a square matrix function that satisfies the matrix equation \( \dot{X} = A(t)X \) and \( \det \Phi(t) \neq 0 \) for all \( t \). Then there is a square constant matrix \( M \) such that \( \Phi(t + \omega) = \Phi(t)M \) for all \( t \) and the eigenvalues \( \mu \) of \( C := \Phi^{-1}(0)\Phi(\omega) \) are called the Floquet multipliers of the Floquet system.
   a. If \( \mu = -1 \) is a Floquet multiplier, prove there is a nontrivial solution with period 2\( \omega \).
   b. Find the Floquet multipliers for the Floquet system
   \[
   \dot{x} = \begin{bmatrix} 1 - \cos t & 0 \\ \cos t & -1 \end{bmatrix} x.
   \]

3. Determine the stability of all equilibrium points of the system
   \[
   \begin{align*}
   \dot{x} &= -x - y^2 \\
   \dot{y} &= -\frac{1}{2}y + 2xy.
   \end{align*}
   \]
   Identify a family of positively invariant sets.

4. This question has two parts.
   a. State the Poincare-Bendixson Theorem.
   b. Use it to show that the system
   \[
   \begin{align*}
   \dot{x} &= x - y + x^3 \\
   \dot{y} &= x + y - y^3
   \end{align*}
   \]
   has a periodic solution. Here the only equilibrium point is \((0, 0)\).
Do three out of four problems below. Clearly indicate in the following boxes which three problems have to be graded, otherwise problems 1, 2, and 3 will be used for grading.

1. Let $\Omega$ be the rectangular domain \{(x, y) : a < x < b; c < y < d\}.

Let $u(x, y) \in C^4(\bar{\Omega})$ be a classical solution of the problem

\[
\begin{align*}
    u_{xxxx} + u_{yyyy} &= -u^{2n+1} \quad \text{in } \Omega, \quad n = 1, 2, 3, \ldots \\
    u(x) &= f(x, y) \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g(x, y) \quad \text{on the boundary } \partial \Omega,
\end{align*}
\]

where $\nu = \nu(x, y)$ is the outward normal vector to the boundary $\partial \Omega$.

Prove that the above solution $u(x, y)$ is unique.

*Hint: You may use Poincaré’s inequality for the functions $w(x, y)$, $w_x(x, y)$ and $w_y(x, y)$ such that $w = w_x = 0$ when $x = a$ and $x = b$, and $w = w_y = 0$ when $y = c$ and $y = d$."

2. Let $u(x) \in C^2(\mathbb{R}^n)$ be a classical solution of the equation $\Delta u = 0$ in $\mathbb{R}^n$, for $n \geq 2$, such that derivatives $u_{x_i} \geq C > -\infty$ for $i = 1, 2, \ldots, n$.

Prove that $u(x) = B + \sum_{i=1}^n b_i x_i$ for some constants $B, b_1, \ldots, b_n$.

*Hint: You may use Liouville Theorem for harmonic functions.*

3. Let $D = U \times (0, \infty)$, where $U$ is a bounded domain in $\mathbb{R}^n$ with $n \geq 1$.

Let $u = u(x, t) \in C^{2,1}_{x,t}(\bar{D})$ be a classical solution of the following initial, boundary value problem for sine-Gordon-type equation

\[
    u_t - \beta \Delta u = \alpha \sin u \quad \text{in } D,
\]
\[ u(x, 0) = u_0(x) \quad \text{on } U, \]
\[ u(x, t) = f(x, t) \quad \text{on } \partial U \times [0, \infty), \]
where \( \alpha \in \mathbb{R} \), and \( \beta > 0 \) are constants.

Prove that this solution \( u(x, t) \) is unique.

*Hint: You may use energy method.*

4. Let \( u(x, t) \) be a classical solution of the Cauchy problem for the wave equation:
\[ u_{tt} - \Delta u = B(u_t, |\nabla u|) \quad x \in \mathbb{R}^n, \ t > 0, \]
with initial data
\[ u(x, 0) = A = \text{constant} \quad \text{and} \quad u_t(x, 0) = 0, \quad \text{for all } |x| \leq R, \]
where \( B \) is a function from \( \mathbb{R}^2 \) to \( \mathbb{R} \), and the positive number \( R \) is fixed.

Assume that there exists a constant \( C > 0 \) such that
\[ |B(z, \xi)| \leq C(|z| + |\xi|) \quad \forall (z, \xi) \in \mathbb{R}^2. \]

Prove that
\[ u(x, t) = A, \]
for all \( (x, t) \) such that \( |x| < R - t \) and \( 0 \leq t \leq R. \)

*Hint: You may use the energy\( E(t) = \int_{\{|x|<R-t\}} u_t^2(x, t) + |\nabla u(x, t)|^2 dx. \)