August 2018 ODE/PDE Preliminary Examination

Part I: ODE. Do 3 of the following 4 problems. You must clearly indicate which 3 are to be graded. Problems 1, 2, and 3 will be graded if no indication is given. Strive for clear and detailed solutions.

1. Consider $\ddot{x} = f(x, \dot{x})$, where $f : \mathbb{R}^{2n} \to \mathbb{R}^n$ is locally Lipschitz continuous on $\mathbb{R}^{2n}$ with Lipschitz constant K. Prove that for every $(x_0, z_0) \in \mathbb{R}^{2n}$, there exists $T > 0$ such that solutions exist on $(-T, T)$.

2. Show that
$$\Phi(t,0) = \begin{bmatrix} e^t & \frac{e^t+\sin t-\cos t}{2} \\ 0 & 1 \end{bmatrix}$$
is the fundamental matrix solution of the linear system
$$\dot{x} = A(t)x(t)$$
with
$$A(t) = \begin{bmatrix} 1 & \cos t \\ 0 & 0 \end{bmatrix}.$$
Find $\Phi(t, \tau)$ and use it to solve the nonhomogenous linear system
$$\dot{x} = A(t)x(t) + b(t), \quad x(0) = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$
where $A(t)$ is given above and $b(t) = \begin{bmatrix} 1 & e^t \end{bmatrix}^T$.

3. Consider the system
$$\begin{aligned} \dot{x} &= 4y^3 - xy^2 \\ \dot{y} &= -3x + x^2 y. \end{aligned}$$
   a. Find all the equilibrium points.
   b. Apply the LaSalle invariance principle to determine the stablilty of the origin.

4. Use the Bendixson-Dulac Theorem to show that the system
$$\begin{aligned} \dot{x} &= y^3 \\ \dot{y} &= -x - y + x^2 + y^4 \end{aligned}$$
has no peroidic orbits.

Theorem (Bendixson-Dulac Theorem) *Assume there is a continuously differentiable function $\alpha(\cdot,\cdot)$ on a simply connected domain $D \subset \mathbb{R}^2$ such that the*
$$\operatorname{div}\left[\alpha(x,y)F(x,y)\right]$$
is either always positive or always negtive on D. Then the system
$$\dot{\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = F(x,y)$$
does not have a peroidic orbit in D.

August 2018. **PRELIMINARY EXAMINATION**

Partial Differential Equations

Do three out of four problems below. Clearly indicate in the following boxes which three problems have to be graded, otherwise problems 1, 2, and 3 will be used for grading.

1. Let U be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary∂U.

Assume $u(x) \in C^2(\bar{U})$ be a classical solution of the problem

$$\Delta u = f(u) \quad \text{in } U,$$
$$u(x) = g(x) \quad \text{on the boundary } \partial U,$$

where function $f(\tau)$ is the differentiate and $\frac{df}{d\tau} \geq 0$.

Prove that if solution of the above problem exists then it is unique.

2. Let $u(x) \in C^2(\mathbb{R}^n)$ $n \geq 2$ be a classical solution of the Laplace equation

$$\Delta u = 0 \text{ in } \mathbb{R}^n.$$

Assume that $u(x) \geq e^{\alpha x_1} \sin(\alpha x_2)$, and $u(0) = 0$.

Prove that $u(x, y) = e^{\alpha x_1} \sin(\alpha x_2)$.

3. Let $D = U \times (0, \infty)$, where $U \subset \mathbb{R}^n$ $n \geq 1$ is a bounded domain with smooth boundary ∂U.

Let $u = u(x, t) \in C^{2,1}_{x,t}\big(\bar{U} \times [0, \infty)\big)$ be a classical solution of the following initial value problem,

$$u_t - \beta \Delta u = 0 \text{ in } D,$$

$$u(x, 0) = u_0(x) \quad \text{on } U,$$

$$u(x,t) = f(x,t) \quad \text{on } \partial U \times [0,\infty),$$

where $\beta = const > 0$, $|fx,t| \leq C < \infty$, and $\lim_{t\to\infty} f(x,t) = 0$.

Prove that if in closure $\bar{U}$ the gradient of the solution of the above problem satisfies inequality $|\nabla u| \leq C < \infty$, uniformly for all (x,t) then

$$\lim_{t\to\infty} \int_U u^2(x,t)dx = 0.$$

4. Consider the Cauchy problem for the wave equation:

$$u_{tt} = u_{xx}, \;\; x \in \mathbb{R}, \;\; t > 0,$$

with initial data

$$u(x,0) = g(x) \quad \text{and} \quad u_t(x,0) = h(x).$$

Prove the following:

a) If $g(x) \in C^2(-\infty,\infty)$, and $h(x) \in C^1(-\infty,\infty)$, then solution of the Cauchy problem exists.

b)If $|g(x)| \leq 1$ and $|h(x)| \leq \exp(-|x|)$ then solution of the Cauchy problem

$$|u(x,t)| \leq 2.$$