ODE Preliminary Exam May 2019

SOLVE ALL FOUR PROBLEMS.

1 Consider the following homogeneous IVP:

$$x'(t) = f(x(t)); \quad x(0) = x_0 \tag{1}$$

where $x_0 \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz continuous function. Suppose that there exist $R > 0$ and $c > 0$ such that $\forall \, \|x\| \geq R$,

$$\|f(x)\| \leq c\sqrt{1 + \|x\|}.$$

Prove that a unique solution $x(t)$ to (1) exists for all $t > 0$. State any theorem you use.

2 An $n \times n$ matrix A is called normal if $A^T A = A A^T$. Some examples of normal matrices are orthogonal matrices, symmetric matrices, and skew-symmetric matrices. Prove that if A is normal, then e^A is normal as well.

Hint: Write A as the sum of a symmetric and a skew-symmetric matrix.

3 Consider the system:

$$\dot{x} = x + y^2$$
$$\dot{y} = -y + xy.$$

(i) Find all the equilibrium points for the system.

(ii) Find the stable eigenspace E^s and the unstable E^u for the equilibrium point $(0, 0)$.

(iii) Construct successive approximate solutions $(x_i(t), y_i(t)), \quad i = 0, 1, 2,$ to find the second order approximation to the stable manifold at $(0, 0)$.

4 Consider the following system:

$$\dot{x} = x\,(a - x - y)$$
$$\dot{y} = y(-3\,a + x),$$

where $a > 0$ is a parameter.

(i) Find all equilibrium points and determine their type.

(ii) Does the system have any periodic (closed) orbits? Give reasons for your answer.

May 2019. **PRELIMINARY EXAMINATION**

Partial Differential Equations

1. Let U be the following unbounded domain in $\mathbb{R}^2$ represented in the polar coordinates as

$$U = \{(r,\theta) : r > 0, \pi/8 < \theta < \pi/4\}.$$

Let $u \in C^2(\bar{U})$ be a classical solution of the problem

$$\begin{aligned} &\Delta u = 0 \quad \text{in } U, \\ &u = 0 \quad \text{on the boundary } \partial U. \end{aligned}$$

Prove that there exists a number $\lambda > 0$ such that if

$$\lim_{r\to\infty} \Big(r^{-\lambda} M(r) \Big) = 0,$$

where $M(r) = \sup\{|u(x)| : x \in U, |x| = r\}$, then $u \equiv 0$ on U.

Hint: Use polar coordinates with

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta},$$

and a barrier function of the form $\mu r^\alpha \sin(\beta\theta)$.

2. Let U be a bounded domain in $\mathbb{R}^n$, with $n \geq 1$, with smooth boundary ∂U. Let $u = u(x,t) \in C^{2,1}_{x,t}\big(\bar{U} \times [0,\infty)\big)$ be a classical solution of the following initial boundary value problem:

$$u_t - \Delta u = f \cdot \nabla u \quad \text{in } U \times (0,\infty),$$

$$u(x,0) = u_0(x) \quad \text{on } U,$$

$$u(x,t) = 0 \quad \text{on } \partial U \times [0,\infty),$$

where the scalar function u_0 and vector-valued $C^{1,0}_{x,t}$-function $f : \bar{U} \times (0,\infty) \to \mathbb{R}^n$ are given.

Prove that there exists a constant $\varepsilon_0 > 0$ such that if

$$|\text{div} f| \le \varepsilon_0 \text{ in } U \times (0, \infty),$$

then

$$\lim_{t\to\infty} \int_U u^2(x,t)dx = 0.$$

3. Consider the Cauchy problem for the wave equation:

$$u_{tt}(x,t) - u_{xx}(x,t) = t^2, \quad x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = e^{x^2} \quad \text{and} \quad u_t(x,0) = \cos(x), \quad x \in \mathbb{R}.$$

Prove that

$$\lim_{t\to\infty} \frac{u(x,t)}{e^{t^2}} = 1.$$