ODE Preliminary Exam Aug 2019

SOLVE ALL FOUR PROBLEMS.

1. A matrix $P : \mathbb{R}^n \to V$ where V is a strict subspace of $\mathbb{R}^n$ is called a *projection matrix* if and only if $P^2 = P$.

   Prove the following:

   (i) If $x \in \mathcal{R}(P)$, the range space of P, then $P\,x = x$. *Hint*: If $x \in \mathcal{R}(P)$, then there is a $w \in \mathbb{R}^n$ such that $P(w) = x$.

   (ii) For $v \in \mathbb{R}^n$, show that $u = v - P(v) \in \mathcal{N}(P)$, the null space of P.

   (iii) From (i) and (ii), find the eigenvalues of P.

   (iv) Consider the system

   $$\dot{x} = -P\,x; \quad x(0) = x_0,$$

   where P is a projection matrix. Prove that

   $$\forall\, t \geq 0, \quad \|x(t)\| \leq \|x_0\|.$$

2 Examine the stability of the origin for the following system using the center manifold theorem.

$$\begin{aligned}\dot{x}_1 &= -x_2 + 3\,x_1\,y + 3\,x_2\,y \\ \dot{x}_2 &= x_1 - x_1\,y + 2\,x_2\,y \\ \dot{y} &= -y + x_1^2\end{aligned}$$

3 Consider the system

$$\dot{x} = -\nabla\,F(x),$$

where ∇f indicates the gradient of the real valued function F. Suppose the origin $0 \in \mathbb{R}^n$ is a cricial point for F. Prove that

   (i) the origin is an asympotically stable node if the origin is a local minimum, and an unstable node if the origin is a local maximum.

   (ii) if a $p \in \mathbb{R}^n$ is a positive limit point of a solution $\phi(\cdot, x_0)$, then p is an equilibrium point for the system.

4 Consider the system:

$$\dot{x}_1 = -\sin x_1 + x_2$$
$$\dot{x}_2 = x_1 - \sin x_2.$$

(i) Find all equilibrium points.

(ii) Does this system possess any periodic (closed) orbits?

August 2019. **PRELIMINARY EXAMINATION**

Partial Differential Equations

Do all three problems below.

1. Let $u \in C^2(\mathbb{R}^2)$ be a classical solution of the equation $\Delta u = 0$ in $\mathbb{R}^2$. Assume that $u(x,y)$ is monotone in the y variable, and

$$u(0,0) = 0, \quad u(1,0) = 1, \quad u(0,1) = 2.$$

Prove that

$$u(x,y) = x + 2y \quad \text{ for all } x, y \in \mathbb{R}.$$

Hint: You may use Liouville's theorem for the function $v = u_y$.

2. Let $D = U \times (0,\infty)$, where U is a bounded domain in $\mathbb{R}^n$, for $n \geq 1$, with C^1-boundary.

Let $u = u(x,t) \in C^{2,1}_{x,t}(\bar{D})$ be a classical solution of the following initial, boundary value problem

$$\begin{aligned} t^\alpha u_t - \Delta u &= u \quad \text{in } D, \\ u(x,0) &= u_0(x) \quad \text{on } U, \\ \frac{\partial u}{\partial \nu} &= f(x,t) \quad \text{on } \partial U \times (0,\infty), \end{aligned}$$

where α is a constant, ν is the outward normal vector to the boundary ∂U, the functions u_0 and f are given.

(a) Prove that the above solution u is unique when $\alpha < 1$.

(b) Find an example for the non-uniqueness of u in case $\alpha = 1$.

3. Let $u(x,t)$ be a classical solution of the Cauchy problem for the wave equation:

$$u_{tt} - u_{xx} = f \quad \text{in} \quad \mathbb{R} \times (0,\infty),$$

with initial data

$$u(x,0) = \cos x \quad \text{and} \quad u_t(x,0) = \sin x,$$

where $f(x,t)$ is a continuous function on $\mathbb{R} \times [0,\infty)$ with compact support.

Prove that, there is a number C such that, for any $x \in \mathbb{R}$, there exists $T > 0$ for which

$$u(x,t) = \cos(x-t) + C \qquad \text{for all } t > T.$$