August 2020 ODE Preliminary Examination

Part I: ODE. Do the following 4 problems. Strive for clear and detailed solutions.

1. Consider a prey predatory system modeled by

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1 - ax_2), \\ \dot{x}_2 &= \tfrac{1}{2}x_2\,(x_1 - x_2), \end{aligned} \tag{1}$$

where '$x_1 \propto$ prey population' and '$x_2 \propto$ predator population'. The parameter a is a constant. Let C be the closed curve shown in Fig. 1.

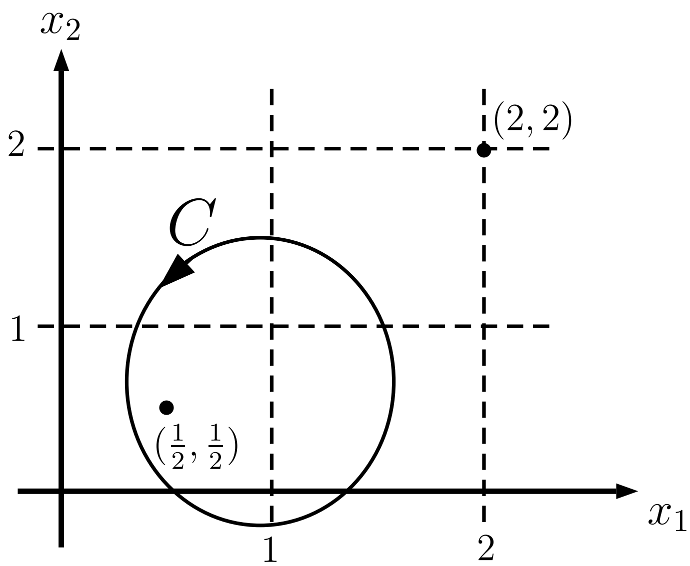

Figure 1: The figure shows the closed curve C.

(a) Find out all equilibrium points of (1) and point out which ones are hyperbolic. Ascertain if the equilibrium points are node, focus, center or saddle.

(b) Let C be as shown in Fig. 1. What is the index of C for $a = 1$ and for $a = -\frac{1}{2}$.

(c) Argue why, for a=1, C cannot be a closed orbit of (1).

2. Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= (x_1x_2 - 1)\,x_2^3 \,+\, (x_1x_2 - 1 + x_1^2)\,x_2, \end{aligned} \tag{2}$$

(a) Show that $x = 0$ is the unique equilibrium point.

(b) Show by linearization that $x = 0$ is asymptotically stable.

(c) Show that

$$\Gamma \;=\; \{x \in R^2 \mid x_1 x_2 \geq 2\} \tag{3}$$

is a positively invariant set.

(d) Is $x = 0$ globally, asymptotically stable?

3. Euler's equations for a rotating rigid spacecraft are given by

$$\begin{aligned} J_1\dot{\omega}_1 \;&=\; (J_2 - J_3)\,\omega_2\,\omega_3 \;+\; u_1, \\ J_2\dot{\omega}_2 \;&=\; (J_3 - J_1)\,\omega_3\,\omega_1 \;+\; u_2, \\ J_3\dot{\omega}_3 \;&=\; (J_1 - J_2)\,\omega_1\,\omega_2 \;+\; u_3, \end{aligned} \tag{4}$$

where ω_1, ω_2, ω_3 are components of the angular velocity vector ω along the principal axes, u_1, u_2, u_3 are the torque inputs applied about the principal axes and J_1, J_2 and J_3 are the principal moments of inertia.

(a) Show that with $u_1 = u_2 = u_3 = 0$, the origin $\omega = 0$ is stable. Is it asymptotically stable?

(b) Suppose that the torque inputs apply feedback

$$u_i \;=\; -k_i\,\omega_i$$

where $k_i > 0, i = 1, 2, 3$. Show that the origin is globally asymptotically stable.

4. We are looking at a pendulum where a bob is hanging from a flexible rod. The rod is modeled by a spring and a damper (see Fig. 2).

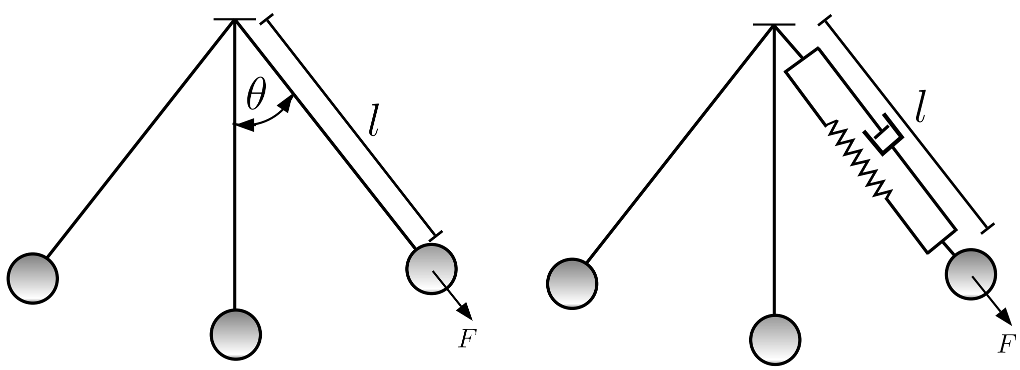

Figure 2: The figure shows a pendulum whose rod is modeled by a spring/damper.

The following is a description of all the parameters:

- μ : spring constant for the rod
- b : damping constant for the rod
- l : variable length of the rod

- θ : angle subtended by the rod (see Fig. 2)
- m : mass of the bob
- g : acceleration due to gravity
- k : coefficient of friction for the pendulum
- F : external force acting along the direction of the rod, assume constant in this problem

The pendulum equation is given by

$$ml\ddot{\theta} + kl\dot{\theta} + mg\sin\theta = 0.$$

The rod equation is given by

$$m\ddot{l} + b\dot{l} + \mu l = F.$$

Let us define

$$x_1 = l,\ x_2 = \dot{l},\ x_3 = \theta,\ x_4 = \dot{\theta}$$
$$\mathbf{X} = (x_1, x_2, x_3, x_4)^T$$

(a) Write down
$$\dot{\mathbf{X}} = F(\mathbf{X}). \tag{5}$$
I want you to write down F.

(b) Where are the equilibrium points of (5)?

(c) If $\mathbf{X}_0$ is an equilibrium point of (5), expand $F(\mathbf{X})$ using Taylor's series about the point $\mathbf{X}_0$.

(d) Define
$$Y = \mathbf{X} - \mathbf{X}_0,$$
so that we have
$$\dot{Y} = AY. \tag{6}$$
For different equilibrium points of the pendulum/rod equation, write down the characteristic polynomial of A.

(e) Assume $k = 0$, i.e. the pendulum is frictionless. We assume that μ, m, b, g, F are all positive constants. Discuss the phase diagram of (6) in the neighborhoods of the equilibrium points.

(f) How does the pendulum oscillation frequency depend on the external force on the rod?

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PRELIMINARY EXAMINATION

Partial Differential Equations

Do all three problems below.

1. Let U be the square $\{(x,t) \in \mathbb{R}^2 : 0 < x < 1, 0 < t < 1\}$ with boundary $\Gamma = \partial U$. Let $\Gamma_0 = [0,1] \times \{1\}$ and $\Gamma_1 = \Gamma \setminus \Gamma_0$.

Assume $u(x,t) \in C^2(U) \cap C(\bar{U})$ and L is the following degenerate operator

$$Lu = 2\frac{\partial u}{\partial t} - (1-t)^\lambda \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2},$$

where $\lambda > 0$ is a constant.

(a) Prove the following maximum principle: If $Lu \leq 0$ in U then

$$\max_{(x,t)\in\bar{U}} u(x,t) \leq \max_{(x,t)\in\Gamma} u(x,t).$$

(b) Prove that if $\lambda > 1$, $Lu \geq 0$ in U, and $u(x,t) \leq 0$ on Γ_1, then

$$u(x,t) \leq 0, \text{ for all } (x,t) \in \bar{U}.$$

(Hint: One can use a barrier function $w = (1-t)^{-\alpha}$ for some appropriate $\alpha > 0$.)

2. Consider the Cauchy problem for the non-linear wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) = -2u(x,t)e^{-u^2(x,t)}, \ x \in \mathbb{R}^n, \ t > 0, \tag{2.1}$$

with initial data

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = \sqrt{2}. \tag{2.2}$$

Let $u(x,t)$ be a classical solution of the Cauchy problem (2.1) and (2.2).

Given $R > 0$. For $t \geq 0$, let

$$I(t) = \int_{B(0,R-t)} \left\{ \frac{1}{2} \left(|u_t(x,t)|^2 + |\nabla u(x,t)|^2 \right) - e^{-u^2(x,t)} \right\} \mathrm{d}x.$$

Here, $B(0,r)$ denotes the ball $\{x \in \mathbb{R}^n : |x| < r\}$.

Prove that there is a positive constant C such that, for all $0 \leq t \leq R$,

$$I(t) \leq C[R^n - (R-t)^n].$$

(Hint: you may use the following formula

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{B(0,g(t))} f(x,t)\mathrm{d}x = g'(t) \cdot \int_{\partial B(0,g(t))} f(x,t)\mathrm{d}s + \int_{B(0,g(t))} \frac{\partial f(x,t)}{\partial t}\mathrm{d}x.)$$

3. Let $u \in C^1(\mathbb{R} \times (0,\infty)) \cap C(\mathbb{R} \times [0,\infty))$ solve the following Cauchy problem:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + e^t u = 0 \text{ in } \mathbb{R} \times (0,\infty),$$
$$u(x,0) = e^{x^2} \text{ in } \mathbb{R}.$$

Prove that

$$\lim_{t \to \infty} u(x,t) = 0 \text{ for any } x \in \mathbb{R}.$$