August 2021 ODE Preliminary Examination

Part I: ODE. Strive for clear and detailed solutions.

1. (a) Using Lyapunov methods to determine the stability of all equilibrium points of the system (1).

   (b) For the stable equilibrium point, determine whether or not it is globally asymptotically stable.

$$\dot{x} = xy^2 - \frac{1}{2}x^3, \ \dot{y} = -\frac{1}{2}y^3 + \frac{1}{5}x^2 y. \tag{1}$$

   Hint: using a quadratic Lyapunov-function $V = ax^2 + by^2$, for suitable constants a and b.

2. Consider the polar coordinate system

$$\dot{r} = r\,(1 - r^2) + \mu\, r\, \cos\theta, \qquad \dot{\theta} = 1, \tag{2}$$

   (a) when $\mu = 0$, prove that there exists a stable limit cycle at $r = 1$.

   (b) show that a closed orbit still exits for $\mu > 0$, as long as μ is sufficiently small. Hint: construct a trapping region, then apply the Poincare-Bendixsion theorem.

3. Consider the system

$$\frac{d^2x}{dt^2} = x - x^3. \tag{3}$$

   (a) Find all the equilibrium points and classify them (center, saddle, node, focus and their stability).

   (b) Find a first integral $F(\dot{x}, x) = C$, where C is an arbitrary constant.

   (c) Sketch the phase portrait. Hint: draw $F(\dot{x}, x) = 0$ and $F(\dot{x}, x) = 1$, then use nullclines to determine the direction of the flow.

August 2021. **PRELIMINARY**

Partial Differential Equations

Do all the three problems below.

1. Assume that $f \in C^{2,1}_{x,t}(\mathbb{R}^n \times [0,\infty))$ has compact support on $\mathbb{R}^n \times [0,\infty)$. Define $u(x,t)$ by

$$u(x,t) \triangleq \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,dyds,$$

where $\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ for $x \in \mathbb{R}^n, t > 0$, is a fundamental solution to heat equation. Prove that

$$\begin{aligned}
&(\partial_t u - \Delta_x u)(x,t) \\
&= \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \partial_t f(x-y,t-s)\,dyds \\
&\quad + \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy - \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \Delta_x f(x-y,t-s)\,dyds
\end{aligned}$$

holds in complete details justifying the hypothesis of Lebesgue's dominated convergence theorem if you use it.

2. Suppose that $\nu > 0$, $u \in C^2(\mathbb{R} \times (0,\infty))$ has compact support in both x and t and solves the initial value problem for wave equation

$$\begin{cases} u_{tt} - \nu u_{xx} = 0 & \text{on } \mathbb{R} \times (0,\infty), \\ u = g \text{ and } u_t = h & \text{on } \mathbb{R} \times \{0\} \end{cases}$$

Suppose that g and h are both in C^1 and have compact support. Let kinetic and potential energy be defined respectively by

$$k(t) \triangleq \frac{1}{2} \int_{\mathbb{R}} u_t^2(x,t)\,dx \text{ and } p(t) \triangleq \frac{\nu}{2} \int_{\mathbb{R}} u_x^2(x,t)\,dx.$$

(a) Prove that $k(t) + p(t) =$ constant for all t.

(b) For simplicity assume that $\nu = 1$, and then prove that $k(t) = p(t)$ for all large enough times t.

3. Derive the solution to the following equation via characteristics: $u = u(x_1, x_2) \in \mathbb{R}$ that solves

$$x_1 u_{x_1} + x_2 u_{x_2} = 2u, \quad u(x_1, 1) = g(x_1),$$

where g is assumed to be smooth. In your proof, denote the spatial domain as simply $dom(u)$.