January 2021 ODE Preliminary Examination

Part I: ODE. Strive for clear and detailed solutions. You can use a simple (non graphing) calculator if you need.

1. In 1961, Fitzhugh proposed the following dynamics to model the spiking activities of neurons:

$$\begin{aligned} \dot{v} &= v - \frac{v^3}{3} - w + 0.5 \\ \dot{w} &= \frac{1}{12.5}(v - 0.7w + 0.8). \end{aligned} \tag{1}$$

You are told that one of the equilibrium points of (1) is

$$v^* = -0.911325,\ w^* \ = ?? \tag{2}$$

(a) Calculate w^* so that (v^*, w^*) is indeed an equilibrium point of (1).

(b) Show that there is a cubic polynomial $p(v)$ in v such that roots of $p(v)$ will describe the equilibrium point of (1). For full credit, write down the cubic polynomial in the form $p(v) = v^3 + av^2 + bv + c$.

(c) Since v^* is a root of $p(v)$, factor theorem states that $v - v^*$ is a factor of $p(v)$. Show that the other two roots of $p(v)$ are complex with positive real parts.

(d) By linearizing (1) around the equilibrium point (2), determine if (2) is a focus, node, saddle, stable or unstable.

(e) We now want to determine using Poincare-Bendixon criterion if (1) has a periodic orbit in the (v, w) plane.

i. State the Poincare-Bendixon criterion by defining M to be a closed and bounded subset in the (v, w) plane. Under what condition would M contain a periodic orbit of (1).

ii. For convenience, we have plotted the isoclines of (1) in Fig. 1. We would like to construct M in the form of a box shown in Fig. 2

$$v_{min} \le v \le v_{max},\ w_{min} \le w \le w_{max}.$$

Sketch M on Fig. 1 and argue that the Poincare-Bendixon criterion is satisfied. You should argue that the vector fields point in the right direction.

Remark: For full credit, I need to see the corner points of M in relation to the isoclines. You can pick numerical values of v_{min}, v_{max}, w_{min}, w_{max}. A sketch of M gives partial credit.

Remark: Indeed (1) is called Fitzhugh-Nagumo equation and has a stable periodic orbit inside M.

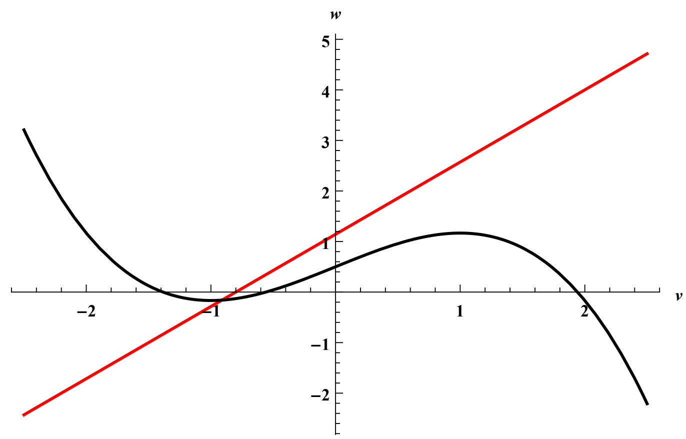

Figure 1: The two isoclines of the dynamics (1).

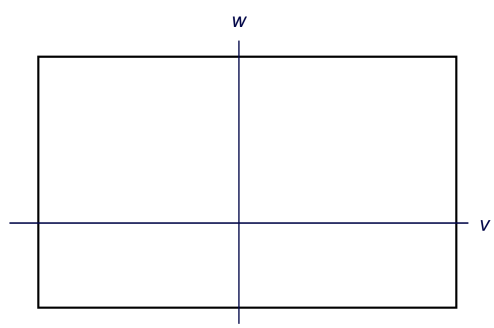

Figure 2: The box M sketched in this figure is not to scale. You need to determine the size of the box by sketching this box on Fig. 1. For full credit, I need to see the coordinates of the corner points.

2. A unit mass $M = 1$ is connected to a pair of springs with an unstretched length of 1 unit, $k = 1$ for both springs, as shown in Fig. 3. For this problem we are assuming zero gravity. Choose positive x and y coordinates as indicated in the Fig. 3.

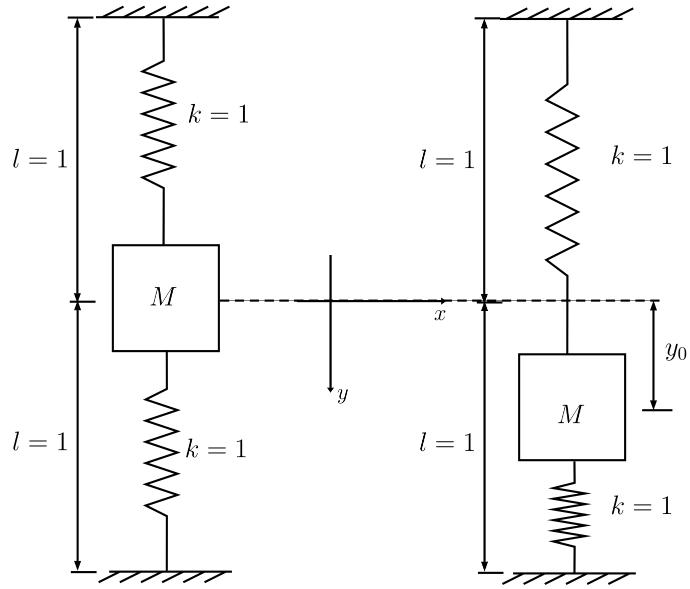

Figure 3: In this figure, the mass M is pulled down.

(a) The mass M is pulled down (see Fig. 3) by an amount $y = y_0$ and released at zero initial velocity. Write down the motion dynamics of the mass in y coordinate. Indicate the equilibrium point of the dynamics and the nature (stable/unstable, saddle, node, center, etc) of this equilibrium point.

(b) The mass M is pulled to the right by an amount $x = x_0$ and released at zero initial velocity (see Fig. 4).

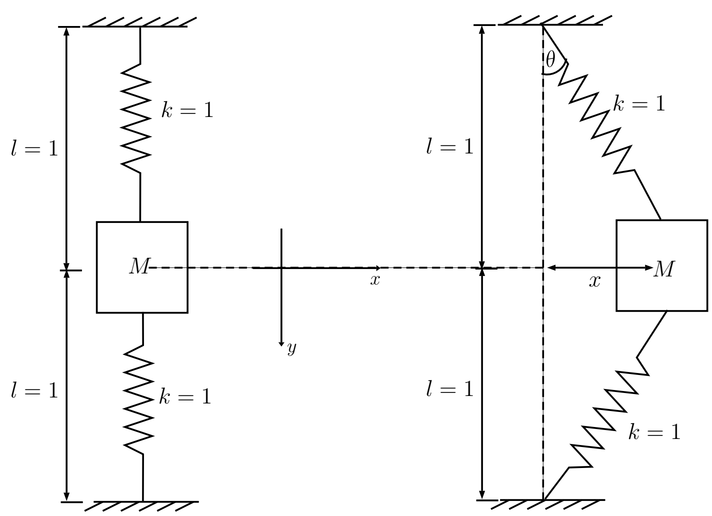

Figure 4: In this figure, the mass M is pulled to the right.

i. Write down the motion dynamics of the mass in the θ coordinate (assume small θ and write $x = \tan\theta \approx \theta$). Use Taylor series of trigonometric functions about $x = 0$ to approximate the dynamics up to cubic terms.

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$$
$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$$
$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

Show that for small angle θ the motion dynamics is given by

$$\ddot{\theta} \;=\; -\theta^3. \tag{3}$$

Hint: Write down the dynamics in the x coordinate and then change variable to θ for small θ.

ii. Indicate the equilibrium point(s) of the dynamics (3) and by linearization, if necessary, ascertain the nature of the equilibrium, if it is hyperbolic or nonhyperbolic.

iii. Using Poincare-Bendixon criterion show that there is a periodic solution for the dynamics (3).

Hint: Use $V(x_1, x_2) = \frac{1}{2}x_1^4 + x_2^2$ and a region $U = \{c_1 \leq V(x_1, x_2) \leq c_2\}$, where $c_2 > c_1 > 0$. The variables x_1, x_2 need to be appropriately defined using θ and $\dot{\theta}$ coordinates.

January 2021.

PRELIMINARY EXAMINATION

Partial Differential Equations

Do all three problems below.

1. Let $L > 0$ be fixed and U be the rectangle

$$\left\{(x, y) \in \mathbb{R}^2 : \frac{L}{2} < x < L, 0 < y < L\right\}.$$

Denote the boundary of U by Γ, $\Gamma_0 = (L/2, L) \times \{L\}$, and $\Gamma_1 = \Gamma \setminus \Gamma_0$.

Define the operator

$$\mathcal{L}u = x(L-y)^{-\lambda}\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2},$$

where $\lambda \geq 0$ is a constant.

(a) Let $u(x, y) \in C^2(U) \cap C(\bar{U})$ and $\delta \in (0, L)$, denote

$$U_\delta = (L/2, L) \times (0, L - \delta).$$

Prove the following maximum principle in U_δ : If $\mathcal{L}u \leq 0$ in U_δ then for any $(x, y) \in \overline{U_\delta}$

$$u(x, y) \leq \max_{(x,y)\in\partial U_\delta} u(x, y).$$

(b) Assume $\lambda > 2$ and $L < 2^{\frac{1}{2-\lambda}}$. Suppose $u \in C^2(U) \cap C(\bar{U})$, $\mathcal{L}u \leq 0$ in U, and $u \leq 0$ on Γ_1. Prove that

$$\max_{(x,y)\in\bar{U}} u(x, y) \leq 0.$$

(Hint: One can use a barrier function $w = \varepsilon(L - y)^{-\alpha}$ for some appropriate $\varepsilon, \alpha > 0$.)

(c) Let λ and L be as in (b). Prove that if $u_1, u_2 \in C^2(U) \cap C(\bar{U})$ satisfy

$$\mathcal{L}u_1 = \mathcal{L}u_2 \text{ in } U \text{ and } u_1 = u_2 \text{ on } \Gamma_1,$$

then $u_1 = u_2$ on U.

2. Let $u(x,t)$, for $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, be a C^2-function on $\mathbb{R}^n \times [0,\infty)$ that solves the Cauchy problem for the non-linear wave equation:

$$u_{tt} - \Delta u - |u_t|^2 + |\nabla u(x,t)|^2 = 0 \text{ in } \mathbb{R}^n \times (0,\infty),$$

with initial data

$$u(x,0) = g(x) \quad \text{and} \quad u_t(x,0) = h(x) \text{ for } x \in \mathbb{R}^n,$$

where g and h are given functions.

Given $R > 0$. For $\alpha \in \mathbb{R}$, let

$$I_\alpha(t) = \frac{1}{2}\int_{B(0,R-t)} e^{\alpha u(x,t)} \left(|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right) \mathrm{d}x \text{ for } t \in [0,R].$$

Here, $B(0,r)$ denotes the ball $\{x \in \mathbb{R}^n : |x| < r\}$.

Prove that there exists a number α such that

$$I_\alpha(t) \le I_\alpha(0) \text{ for all } t \in [0,R].$$

(*Hint: you may use substitution $v = f(u)$ for an appropriate function f.*)

3. Let $u(x,t)$, for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $t \in [t_0,\infty)$ with t_0 being a fixed number in $\mathbb{R}$, be a function in $C^1(\mathbb{R}^n \times [t_0,\infty))$ that solves the equation

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial t} = -u^2 \text{ in } \mathbb{R}^n \times (t_0,\infty).$$

Suppose that $u(x,t) > 0$ for all $(x,t) \in \mathbb{R}^n \times [t_0, \infty)$, and there exist $R > 0$ and $C > 0$ such that

$$u(x, t_0) \leq C \text{ for all } |x| > R.$$

Prove, for any $x \in \mathbb{R}^n$, that

$$\lim_{t \to \infty} \Big(t u(x,t) \Big) = 1.$$

(*Hint: Use an appropriate substitution* $v = f(u)$.)