May 2021 ODE Preliminary Examination

Part I: ODE. Complete all of the following questions. Strive for clear and detailed solutions.

1. (a) Find invariant manifolds for the system (1)

$$\dot{x} = x(1 - 4x + y), \ \dot{y} = y(2 + 3x - 2y). \tag{1}$$

   (b) Prove that the system (1) has no limit cycles in the first quadrant.

2. We consider the following equations

$$\dot{x} = \frac{\partial E}{\partial y} + \lambda E \frac{\partial E}{\partial x}, \qquad \dot{y} = -\frac{\partial E}{\partial x} + \lambda E \frac{\partial E}{\partial y}, \tag{2}$$

   with $\lambda \in \mathbb{R}$, $E(x, y) = y^2 - 2x^2 + x^4$.

   (a) Put $\lambda = 0$.

      i. Determine the critical points and their character (center, saddle, stable/unstable node, or stable/unstable focus) by linear analysis.

      ii. Show $E(x, y) =$ constant is invariant in terms of time. Then, sketch the phase-plane for the nonlinear system.

      Hints:

      i. indicate the equilibrium points;

      ii. draw contour plot of $E(x, y) = C$ for $C = 0$, $C < 0$ and $C > 0$;

      iii. using the local stability of the corresponding equilibrium points to determine the contour plot of $E(x, y) = 0$ and $E(x, y) = C < 0$;

      iv. for the contour plot of $E(x, y) = C > 0$, consider the flow of the vector field.

   (b) In the case of $\lambda \neq 0$,

      i. prove that $E(x, y) = 0$ represents an invariant set.

      ii. draw the homoclinic loops β_1 and β_2 containing in the invariant set $E(x, y) = 0$ and prove the existence of β_1 and β_2. The homoclinic loop denotes circular trajectories joining the saddle point (0,0).

      iii. draw phase-plane diagram for the case $\lambda > 0$ and the case $\lambda < 0$.

May 2021. **PRELIMINARY**

Partial Differential Equations

Do all the three problems below.

1. Given a number $k > 0$, let

$$D = \{(x, y) \in \mathbb{R}^2 : 0 < x < \infty, -kx < y < kx\},$$

and define an unbounded domain $U = \mathbb{R}^2 \setminus \bar{D}$. Assume that $u(x, y) \in C^2(U) \cap C(\bar{U})$ is a solution of the following boundary value problem:

$$\Delta u = 0 \quad \text{in } U,$$
$$u(x, kx) = u(x, -kx) = 0 \text{ for all } x \geq 0.$$

For $r > 0$, let $M(r) = \max\left\{|u(x, y)| : (x, y) \in \bar{U},\ \sqrt{x^2 + y^2} = r\right\}$. Prove that if there is a positive number m such that

$$\lim_{r \to \infty} \frac{M(r)}{(\ln r)^m} < \infty,$$

then $u(x, y) = 0$ for all $(x, y) \in U$.

(*Hint: One can write the Laplacian in the polar coordinates* (r, φ), *and construct a barrier function of the form* $v(r, \varphi) = Cr^\lambda \sin(\lambda\varphi)$.)

2. Assume that $f \in L^1_{\text{loc}}((0, \infty))$, $h \in C^2_1(\mathbb{R}^n \times (0, \infty))$ has compact support and $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $n = 2, 3, \ldots$ Find the explicit form of u that solves

$$\begin{cases} \partial_t u - f(t)u = \Delta u + h & \text{on } \mathbb{R}^n \times (0, \infty), \\ \lim_{t \to 0^+} u(x, t) = g(x) & \text{at } \mathbb{R}^n \times \{0\}. \end{cases} \tag{2.1}$$

3. Let U be a domain in $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial U$. Suppose that a function $f : U \times (0, \infty) \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies

$$p\, f(x, t, p, q) \leq -p(p^2 - |q|^2) \text{ for all } (x, t, p, q) \in U \times (0, \infty) \times \mathbb{R} \times \mathbb{R}^n.$$

Given $T > 0$, let $u(x,t) \in C^2(\bar{U} \times [0,T])$ be the classical solution of

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f(x,t,u_t,\nabla u), \ \ (x,t) \in U \times (0,T],$$

with the initial condition

$$u(x,0) = g(x) \quad \text{and} \quad u_t(x,0) = h(x), \text{for } x \in U,$$

and the boundary condition

$$u(x,t) = const. \text{ for } (x,t) \in \Gamma \times (0,T].$$

Prove that

$$\int_U e^{2u(x,t)} \left(|u_t(x,t)|^2 + |\nabla u(x,t)|^2 \right) dx \leq \int_U e^{2g(x)} \left(h^2(x) + |\nabla g(x)|^2 \right) dx$$

for all $t \in (0,T]$.

(*Hint: you may use the substitution* $v(x,t) = \psi(u)$ *for an appropriate function* ψ.)