August 2023 **PRELIMINARY**

Partial Differential Equations

Instructions

Do any six of the seven problems below. You must clearly indicate which six of the seven problems you want us to grade. If you do not, we choose six arbitrarily.

Problems

1. Suppose that $u \in C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ solves the equation $\Delta u = 0$ in $\mathbb{R}^n$. Show that $u \equiv 0$.

2. Suppose u is a smooth solution to $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$. Show that $v(x,t) := x.Du(x,t) + 2tu_t(x.t)$ also solves the heat equation.

3. Prove that there exists at most one solution $u \in C^2(U_T)$ to the following system of equations:

$$\begin{cases} \partial_{tt}u - \Delta u = \frac{\partial u}{\partial x_1} & \text{in } U_T, \\ u = g & \text{on } \partial \bar{U}_T, \\ \partial_t u = h & \text{on } U \times \{t = 0\}, \end{cases}$$

where x_1 is the first component of $x \in \mathbb{R}^n$.

4. Write down an explicit solution for a function u solving the following IVP:

$$\begin{cases} \partial_t u + b\partial_2 u = f(x,t) \text{ in } & \mathbb{R}^2 \times (0, \infty) \\ u(t,0) = g(x) & \text{on } \mathbb{R}^2 \times \{t = 0\}. \end{cases}$$

Here a and b are constant real numbers.

5. Solve the following using characteristics:

$$uu_{x_1} + u_{x_2} = 1, \qquad u(x_1, x_1) = \frac{1}{2}x_1.$$

6. Let H^* be the Legendre transform of a function H. For $H(p) = \frac{||p||^2}{2023}$, find H^*.

7. Use a transformation method to solve the PDE, for $f \in L^2(\mathbb{R}^n)$,

$$-\Delta u + 8u = f.$$

Appendix

- $\mathbb{R}^n$ is the n-dimensional Euclidean space.
- $U_T = U \times (0, T]$.
- IVP stands for initial value problem.
- We recall the Legendre transform of a function H:

$$H^*(p) = \sup_{v \in \mathbb{R}^n} \{p \cdot v - H(v)\}.$$

- For any function $u \in L^1(\mathbb{R}^n)$, we define its Fourier transform as

$$\mathcal{F}(u)(\xi) \triangleq \hat{u}(\xi) \triangleq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) dx,$$

  and its inverse

$$\mathcal{F}^{-1}(u)(x) \triangleq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(\xi) d\xi.$$