May 2022 **PRELIMINARY**

Partial Differential Equations

Do any six of the seven problems below. You must clearly indicate which six of the seven problems you want us to grade. If you do not, then we will grade Questions 1-6.

1. Suppose that $u : \mathbb{R}^n \mapsto \mathbb{R}$.

(a) Give the precise statement of Liouville's theorem; its hypothesis and its implication.

(b) Fix any $p \in [1, \infty)$. Suppose that $u : \mathbb{R}^n \mapsto \mathbb{R}$ satisfies $u \in L^p(\mathbb{R}^n)$ and is harmonic on $\mathbb{R}^n$. Prove or give a counterexample to the claim that $u \equiv 0$ on $\mathbb{R}^n$.

2. For any function $f \in L^1(\mathbb{R})$, we define its Fourier transform as

$$\mathcal{F}(u)(\xi) \triangleq \hat{u}(\xi) \triangleq \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-ix\xi} u(x) dx,$$

and its inverse

$$\mathcal{F}^{-1}(u)(x) \triangleq \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ix\xi} u(\xi) d\xi.$$

Suppose that $u : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{C}$ is in C_c^∞ in both x and t. Derive in detail via Fourier transform the fundamental solution of the heat equation

$$\partial_t u = \partial_{xx} u.$$

3. Fix $x_0 \in \mathbb{R}^n$ and $t_0 > 0$. Suppose that $u \in C^2(\mathbb{R}^n \times (0, \infty))$ solves

$$\partial_{tt} u - \Delta u = 0 \text{ for all } (x, t) \in \mathbb{R}^n \times (0, \infty)$$

and $u \equiv \partial_t u \equiv 0$ on $B(x_0, t_0) \times \{t = 0\}$. Prove that $u \equiv 0$ on a cone

$$B(x_0, t_0 - t) \times [0, t_0] = \{(x, t) : |x - x_0| \leq t_0 - t, 0 \leq t \leq t_0\}.$$

4. Solve the following via characteristics:

$$u\partial_{x_1}u + \partial_{x_2}u = 1, \qquad u(x_1, x_1) = \frac{1}{2}x_1.$$

5. Suppose that $H : \mathbb{R}^n \mapsto \mathbb{R}$ satisfies the following:

$$H \in C^2, \qquad \lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty, \quad \text{and} \quad p \mapsto H(p) \text{ is uniformly convex.}$$

Recall the definition of a weak solution to an IVP of Hamilton-Jacobi equation: given $g : \mathbb{R}^n \mapsto \mathbb{R}$ that is Lipschitz, $u : \mathbb{R}^n \times [0, \infty) \mapsto \mathbb{R}$ that is Lipschitz (in both x and t) is a weak solution of the IVP

$$\partial_t u + H(Du) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \tag{5.1a}$$

$$u = g \quad \text{on } \mathbb{R}^n \times \{t = 0\} \tag{5.1b}$$

if

(a) (5.1a) is satisfied for almost every $(x, t) \in \mathbb{R}^n \times (0, \infty)$,

(b) (5.1b) is satisfied,

(c) there exists $C \geq 0$ such that

$$u(x+z, t) - 2u(x, t) + u(x-z, t) \leq C(1 + \frac{1}{t})|z|^2 \quad \forall\, x, z \in \mathbb{R}^n, t > 0.$$

Suppose that u^1, u^2 are two weak solutions of the IVP of Hamilton-Jacobi equation; i.e., for both $i \in \{1, 2\}$,

$$\partial_t u^i + H(Du^i) = 0 \qquad \text{in } \mathbb{R}^n \times (0, \infty),$$

$$u^i = g^i \qquad \text{on } \mathbb{R}^n \times \{t = 0\},$$

with potentially distinct g^1, g^2 that are both Lipschitz. Prove the L^∞-contraction inequality: for all $t > 0$,

$$\sup_{x\in\mathbb{R}^n} |u^1(x, t) - u^2(x, t)| \leq \sup_{x\in\mathbb{R}^n} |g^1(x) - g^2(x)|.$$

6. Recall Airy's equation: $u(x,t)$ for $x \in \mathbb{R}, t \in \mathbb{R}$ that satisfies

$$\partial_t u + \partial_{xxx} u = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}.$$

(a) Give a definition of what it means for a PDE to be dispersive. Prove or disprove that Airy's equation is dispersive.

(b) Suppose that $u \in C_t^1 \cap C_x^3$, $\partial_t \partial_x u$ is also continuous, and u has compact support K in x that is valid for all $t > 0$; i.e., for all $t > 0$, $u(x,t) = 0$ if $x \notin K$. Suppose that u satisfies the damped Airy's

$$\partial_t u + u + \partial_{xxx} u = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}.$$

Suppose that initial data $u(0,t)$ that satisfies $\|\partial_x u(0)\|_{L^2(\mathbb{R})}^2 < \infty$ is given to us. Prove the following exponential decay for $t > 0$:

$$\|\partial_x u(t)\|_{L^2(\mathbb{R})}^2 = \|\partial_x u(0)\|_{L^2(\mathbb{R})}^2 e^{-2t}.$$

7. Prove the following interpolation inequality in Hölder norm: for any $0 < \beta < \gamma \leq 1$:

$$\|u\|_{C^{0,\gamma}(U)} \leq \|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}.$$