May 2023 **PRELIMINARY**

Partial Differential Equations

Do any six of the seven problems below. You must clearly indicate which six of the seven problems you want us to grade. If you do not, we choose six arbitrarily.

1. Let $U = \mathbb{B}^n(0, 2023)$ an Euclidean ball. Suppose that $u, v \in C^2(\overline{U})$ are harmonic functions on U and $u \geq v$. If $u(0) = v(0) = 100$ then show that $u(x) = v(x)$ for all $x \in U$.

2. Let $U \subset \mathbb{R}^n$ be open and bounded. Suppose that $u_1, u_2 \in C^2(\bar{U}_T)$ solve the IVP:

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{on } U_T, \\ u = g & \text{on } \partial U \times [0, T], \end{cases}$$

Prove that if $u_1(x, T) = u_2(x, T)$ for all $x \in U$, then $u \equiv \tilde{u}$ in U_T.

3. Use the Fourier transform to solve the wave equation

$$\begin{cases} u_{tt} - \Delta u & = 0 \text{ on } \mathbb{R}^n \times (0, \infty), \\ u = g, u_t & = 0 \text{ on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

4. Write down an explicit solution for the following IVP:

$$\begin{cases} \partial_t u + a\partial_1 u + bu = 0 \text{ in } & \mathbb{R}^2 \times (0, \infty) \\ u(t, 0) = g(x) & \text{on } \mathbb{R}^2 \times \{t = 0\}. \end{cases}$$

Here a and b are constant real numbers.

5. Solve the following using characteristics:

$$x_1 u_{x_1} + u_{x_2} + 10x_3 u_{x_3} = 3u, \quad u(x_1, 0, x_3) = g(x_1, x_3).$$

6. Let H^* be the Legendre transform of a convex function H. Let $H(p) = ||p||^2 - 1$. Find H^*.

7. Use separation of variables to find a non-trivial solution u of the PDE

$$u_{x_1}^2 u_{x_1x_1} + 2u_{x_1}u_{x_2}u_{x_1x_2} + u_{x_2}^2 u_{x_2x_2} = 0 \quad \text{in } \mathbb{R}^2.$$

Appendix:

- $\mathbb{R}^n$ is the n-dimensional Euclidean space.
- $U_T = U \times (0, T]$.
- IVP stands for initial value problem.
- We recall the Legendre transform of a function H:

$$H^*(p) = \sup_{v \in \mathbb{R}^n} \{p \cdot v - H(v)\}.$$

- For any function $u \in L^1(\mathbb{R}^n)$, we define its Fourier transform as

$$\mathcal{F}(u)(\xi) \triangleq \hat{u}(\xi) \triangleq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx,$$

  and its inverse

$$\mathcal{F}^{-1}(u)(x) \triangleq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(\xi)d\xi.$$