

Probability and Statistics Preliminary Examination: August 2022

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
 - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
 - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
 - $I(x \in A)$ or $I_A(x)$: indicator function for set A ; takes on the value 1 if $x \in A$ and 0 otherwise.
 - $\mathbb{E}(X)$: expectation of random variable X .
 - $\mathbb{V}(X)$: variance of random variable X .
 - $X \sim N(a, b)$: X has a normal distribution with mean a and variance b .
- Common distributions and other results.

Bin(n, p): $\mathbb{E}(X) = np$, $\mathbb{V}(X) = np(1 - p)$, and pmf and mgf given respectively by

$$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} I(x \in \{0, 1, \dots, n\}), \quad M(t) = (1-p + pe^t)^n$$

Unif(a, b): $\mathbb{E}(X) = (a + b)/2$, $\mathbb{V}(X) = (b - a)^2/12$, and pdf given by

$$f(x) = (b - a)^{-1} I(a < x < b)$$

Pois(λ): $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!} I(x \in \{0, 1, \dots\}), \quad M(t) = \exp\{\lambda(e^t - 1)\}$$

Geometric(p): $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1 - p)/p^2$, and pmf and mgf given respectively by

$$f(x) = p(1 - p)^{x-1} I(x \in \{1, 2, \dots\}), \quad M(t) = \frac{pe^t}{1 - (1 - p)e^t}, \quad t < -\log(1 - p)$$

Negative-Binomial(r, p): $\mathbb{E}(X) = r(1 - p)/p$, $\mathbb{V}(X) = r(1 - p)/p^2$, and pmf and mgf, respectively:

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x I(x \in \{0, 1, \dots\}), \quad M(t) = \left(\frac{p}{1 - (1-p)e^t} \right)^r, \quad t < -\log(1-p)$$

Gamma(α, β): $\mathbb{E}(X) = \alpha\beta$, $\mathbb{V}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0), \quad M(t) = \left(\frac{1}{1 - \beta t} \right)^\alpha, \quad t < 1/\beta$$

Exp(β): Gamma(1, β).

Beta(α, β): $\mathbb{E}(X) = \frac{\alpha}{\alpha + \beta}$, $\mathbb{V}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$, and pdf given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} I(0 < x < 1)$$

Order Statistics: Let $X_{(1)} \leq \dots \leq X_{(n)}$ denote the order statistics from a random sample X_1, \dots, X_n . If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \dots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} [F(x)]^{j-1} [1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)}, X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-1-i} [1-F(x_j)]^{n-j} \\ \times I(-\infty < x_i \leq x_j < \infty)$$

$$f_{X_{(1)}, \dots, X_{(n)}}(x_1, \dots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \leq \dots \leq x_n < \infty)$$

Problems

1. Let X_1, \dots, X_n be iid observations from $\text{Unif}(0, \theta^2)$ with $\theta > 0$.
 - (a) Find the MOME of θ .
 - (b) Let $\hat{\theta}$ be the MOME in (a). Show that $\hat{\theta} \xrightarrow{P} \theta$.
 - (c) Find the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} ?$

2. Suppose that X_1, \dots, X_n are iid random variables from $\text{Exp}(\beta)$. However, we only observe the variables Y_1, \dots, Y_n where $Y_i = I(X_i > 1)$.
 - (a) Based on the sample Y_1, \dots, Y_n , find the MLE of β .
 - (b) Let $\hat{\beta}$ be the MLE in (a). Find the limiting distribution of $\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{D} ?$

3. Let X_1, \dots, X_n be iid observations with pdf $f(x) = 3x^2 I(0 < x < 1)$. Let $Y_n = \min\{X_1, \dots, X_n\}$.
 - (a) Find the cdf of Y_n .
 - (b) Show that $n^{1/3}Y_n \xrightarrow{D} Z$ for some random variable Z and find the cdf of Z .
 - (c) When $\alpha \in (0, 1/3)$, show that $n^\alpha Y_n \xrightarrow{P} 0$.

(**Hint:** You may need the formula $\lim_{n \rightarrow \infty} (1 + a_n/n)^n = \begin{cases} e^a & \text{if } \lim_{n \rightarrow \infty} a_n = a; \\ \infty & \text{if } \lim_{n \rightarrow \infty} a_n = \infty; \\ 0 & \text{if } \lim_{n \rightarrow \infty} a_n = -\infty. \end{cases}$)

4. Let X_1, \dots, X_n be independent random variables from $\text{Exp}(\beta)$. Let $T = \sum_{i=1}^n X_i$.
- (a) Are X_1/T and T independent? Why or why not.
 - (b) Find the conditional pdf of X_1/T given $T = t$.
 - (c) Find the UMVUE of β^2 .
- (**Hint:** You may need the following result. Suppose $X \sim \text{Gamma}(\alpha_1, \beta)$ and $Y \sim \text{Gamma}(\alpha_2, \beta)$ are independent variables. Then $X/(X + Y) \sim \text{Beta}(\alpha_1, \alpha_2)$.)
5. Let X_1, X_2 be iid observations from $\text{Exp}(\theta)$. Consider the following two hypotheses, $H_0 : \theta = 1$ V.S. $H_1 : \theta > 1$.
- (a) Test 1: reject H_0 if $X_1 + X_2 > c_1$. Find the condition on c_1 such that Test 1 will have size α .
 - (b) Test 2: reject H_0 if $\max\{X_1, X_2\} > c_2$. Find the value of c_2 such that Test 2 has size α .
 - (c) Find the power function of Test 2.
 - (d) Which test is more powerful? Briefly discuss the reason.