Probability and Statistics Preliminary Examination: May 2023

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I(x \in A)$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbb{E}(X)$: expectation of random variable X.
  - $\mathbb{V}(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  Bin(n, p)**:** $\mathbb{E}(X) = np$, $\mathbb{V}(X) = np(1-p)$, and pmf and mgf given respectively by

  $$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} I(x \in \{0, 1, \cdots, n\}), \qquad M(t) = (1 - p + pe^t)^n$$

  Poisson(λ)**:** $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

  $$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  Geometric(p)**:** $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

  $$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1 - (1-p)e^t}, \quad t < -\log(1-p)$$

  Negative-Binomial(r, p)**:** $\mathbb{E}(X) = r(1-p)/p$, $\mathbb{V}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

  $$f(x) = \binom{r + x - 1}{x} p^r (1-p)^x I(x \in \{0, 1, \cdots\}), \qquad M(t) = \left(\frac{p}{1 - (1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

Gamma(α, β)**:** $\mathbb{E}(X) = \alpha\beta$, $\mathbb{V}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < 1/\beta$$

Exp(β)**:** Gamma$(1, \beta)$.

Beta(α, β)**:** $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathbb{V}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, and pdf given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0 < x < 1)$$

Order Statistics: Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)}, X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-1-i} [1 - F(x_j)]^{n-j} \times I(-\infty < x_i \leq x_j < \infty)$$

$$f_{X_{(1)}, \ldots, X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \leq \cdots \leq x_n < \infty)$$

Problems

1. Let the joint pdf of X and Y be

$$f(x,y) = \begin{cases} 2 & \text{if } x > 0, y > 0, x + y < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

   Find the pdf of X/Y.

2. Suppose that $X_1, \ldots, X_m$ has a multinomial distribution with pmf given by

$$f(x_1, \ldots, x_m | p_1, \ldots, p_m) = \frac{n!}{x_1! \ldots x_m!} p_1^{x_1} \ldots p_m^{x_m}.$$

Here $0 \le x_i \le n$, $\sum_{i=1}^{m} x_i = n$, and $\sum_{i=1}^{n} p_i = 1$.

(a) Find a likelihood ratio test (LRT) statistic for the hypotheses $H_0 : p_1 = p_2 = \ldots = p_m = \frac{1}{m}$ V.S. $H_1 : H_0$ is false.

(b) Find the asymptotic size α rejection region based on the LRT statistic in (a).

3. Suppose $X_1, \cdots, X_n$ are i.i.d with uniform distribution $Unif(-\theta, \theta)$ with $\theta > 0$.

   (a) Find the MLE $\hat{\theta}$ of θ.

   (b) Find the distribution of $\hat{\theta}/\theta$.

   (c) Using the exact distribution in (2), find the value of m such that $[\hat{\theta}, m\hat{\theta}]$ is a $1-\alpha$ confidence interval for θ.

4. Let $X_1, \ldots, X_n$ be independent random variables from $Bin(1, \theta)$ with $\theta \in (0,1)$. Let $T = \sum_{i=1}^{n} X_i$.
   (a) Find the conditional pmf of X_i given $T = t$.
   (b) Find the UMVUE of $\theta(1-\theta)$.

5. Let $X_1, \dots, X_n$ be i.i.d observations from $Pois(\lambda)$ with $\lambda > 0$. Let us define

$$T = \sum_{i=1}^{n} X_i, \quad W = (1 - 1/n)^T, \quad \tau(\lambda) = e^{-\lambda}.$$

   (a) Show that W is an unbiased estimator of $\tau(\lambda)$.
   (b) Calculate the variance of W.
   (c) Is W a UMVUE? Prove or disprove.