Probability and Statistics Preliminary Examination: May 2024

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I(x \in A)$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbb{E}(X)$: expectation of random variable X.
  - $\mathbb{V}(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  Bin(n, p)**:** $\mathbb{E}(X) = np$, $\mathbb{V}(X) = np(1-p)$, and pmf and mgf given respectively by

  $$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} I(x \in \{0, 1, \cdots, n\}), \qquad M(t) = (1 - p + pe^t)^n$$

  Poisson(λ)**:** $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

  $$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  Geometric(p)**:** $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

  $$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1 - (1-p)e^t}, \quad t < -\log(1-p)$$

  Negative-Binomial(r, p)**:** $\mathbb{E}(X) = r(1-p)/p$, $\mathbb{V}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

  $$f(x) = \binom{r + x - 1}{x} p^r (1-p)^x I(x \in \{0, 1, \cdots\}), \qquad M(t) = \left(\frac{p}{1 - (1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

Gamma(α, β)**:** $\mathbb{E}(X) = \alpha\beta$, $\mathbb{V}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < 1/\beta$$

Exp(β)**:** Gamma$(1, \beta)$.

Beta(α, β)**:** $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathbb{V}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, and pdf given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0 < x < 1)$$

Order Statistics: Let $X_{(1)} \le \cdots \le X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} [F(x)]^{j-1} [1 - F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)}, X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-1-i} [1 - F(x_j)]^{n-j} \times I(-\infty < x_i \le x_j < \infty)$$

$$f_{X_{(1)}, \ldots, X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \le \cdots \le x_n < \infty)$$

Problems

1. Suppose $X_1, \cdots, X_n$ ($n \geq 2$) are i.i.d from $Unif(\theta, 2\theta)$ for some $\theta > 0$.

   (a) Find the MLE of θ.

   (b) Is the MLE unbiased? Prove or disprove.

2. Let X, Y be two independent random variables following $Unif(0,1)$. Find the density function of $\frac{Y}{X+Y}$.

3. Suppose $X_1, \cdots, X_n$ are i.i.d from $Bin(1, \theta)$ for $\theta \in (0,1)$. Let $\overline{X}$ be the sample average.

   (a) Find a function τ such that $\tau(\overline{X}) \xrightarrow{P} \theta(1-\theta)$.

   (b) When $\theta \neq 1/2$, find the limiting distribution of $\sqrt{n}\left(\tau(\overline{X}) - \theta(1-\theta)\right) \xrightarrow{D}$?

   (c) When $\theta = 1/2$, is the convergence in (b) still valid? Discuss the reason.

4. Suppose that $X_1, \dots, X_n$ are i.i.d. observations from the cdf

$$F(x) = \begin{cases} 0 & \text{if } x \le 0; \\ (x/\beta)^3 & \text{if } 0 < x \le \beta; \\ 1 & \text{if } x > \beta. \end{cases}$$

(a) Show that $T = X_{(1)}/\beta$ is a pivotal quantity.

(b) Based on T, find the value of $u = u(X_1, \dots, X_n)$ such that $[0, u]$ is a $1 - \alpha$ confidence interval for β.

5. Suppose that $X_1, \dots, X_n$ are i.i.d. observations from $Beta(\theta, 1)$ with $\theta > 0$. Suppose that you only record the observations $Y_i = I(X_i < 1/2)$ for $i = 1, \dots, n$.

   (a) Find the MLE of θ based on the sample $Y_1, \dots, Y_n$.

   (b) Let $\hat{\theta}$ be the MLE in (a). Find the limiting distribution $\sqrt{n}(\hat{\theta} - \theta) \stackrel{D}{\to}$?

   (c) Based on the result in (b), find an asymptotic $1 - \alpha$ confidence interval for θ.