Probability and Statistics Preliminary Examination: August 2025

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); i.i.d. (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I(x \in A)$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbf{E}(X)$: expectation of random variable X.
  - $\mathbf{Var}(X)$: variance of random variable X.
  - $X \sim N(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  (i) $Bin(n,p)$: $\mathbf{E}(X) = np$, $\mathbf{Var}(X) = np(1-p)$, and pmf and mgf given respectively by

$$f(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}I(x \in \{0,1,\cdots,n\}), \qquad M(t) = (1-p+pe^t)^n$$

  (ii) $Pois(\lambda)$: $\mathbf{E}(X) = \lambda$, $\mathbf{Var}(X) = \lambda$, and pmf and mgf given respectively by

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}I(x \in \{0,1,\cdots\}), \qquad M(t) = \exp\{\lambda(e^t-1)\}$$

  (iii) $Geom(p)$: $\mathbf{E}(X) = 1/p$, $\mathbf{Var}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

$$f(x) = p(1-p)^{x-1}I(x \in \{1,2,\cdots\}), \qquad M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$$

  (iv) $NB(r,p)$: $\mathbf{E}(X) = r(1-p)/p$, $\mathbf{Var}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

$$f(x) = \binom{r+x-1}{x}p^r(1-p)^xI(x \in \{0,1,\cdots\}), \qquad M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

(v) *Gamma*(α, β): $\mathbf{E}(X) = \alpha\beta$, $\mathbf{Var}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < 1/\beta$$

(vi) *Exp*(β): Gamma$(1, \beta)$.

(vii) *Beta*(α, β): $\mathbf{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathbf{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, and pdf given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0 < x < 1)$$

(viii) Order Statistics: Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)},X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j)[F(x_i)]^{i-1}[F(x_j) - F(x_i)]^{j-1-i}[1-F(x_j)]^{n-j} \times I(-\infty < x_i \leq x_j < \infty)$$

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \leq \cdots \leq x_n < \infty)$$

Problems

1. Suppose that $X \sim Exp(\lambda)$ and $\mathbf{E}((X-1)(X-2)) = 1$. Find all the possible values of λ.

2. Suppose that $X_1, \ldots, X_n$ are i.i.d with $N(\mu, \sigma^2)$. Both $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown. Find the UMVUE of μ^2. (**Hint:** $\mu^2 = \mathbf{E}(X^2) - \mathbf{Var}(X)$.)

3. Suppose the random variable Y has a pdf $f_Y(y) = y^{-2} I(y > 1)$. Conditioning on $Y = y$, the random variables $X_1, \ldots, X_n$ are i.i.d $Unif(0, y)$. Let $X_{(n)} = \max_{1 \le i \le n} X_i$. Find the conditional pdf $f_{Y|X_{(n)}}$.

4. Suppose that $X_1, \ldots, X_n$ are i.i.d from the pdf

$$f(x|\theta) = \frac{2\theta(1-x)}{(2x - x^2)^{1-\theta}}, \text{ when } x \in (0,1).$$

   Here $\theta > 0$ is the parameter.

   (a) Find the MLE of θ.

   (b) Let $\hat{\theta}$ be the MLE in (a). Find the limiting distribution $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D}$?

   (c) Consider the hypotheses: $H_0 : \theta = \theta_0 \quad V.S \quad H_1 : \theta \neq \theta_0$. Here $\theta_0 > 0$ is a known value. Use the result in (b) to build an asymptotic size-α test.

   (d) Use the result in (b) to build an asymptotic $1 - \alpha$ confidence interval.

5. Suppose $Y \sim Exp(\theta)$ with $\theta > 0$, and T is a random variable taking values in $[0, \infty)$ with pdf $g(t)$ and cdf $G(t)$. Moreover, Y and T are independent. Define random variables:

$$X = \min(Y, T), \quad Z = \begin{cases} 1 & \text{if } Y \le T, \\ 0 & \text{if } Y > T. \end{cases}$$

   (a) Find the the probability $P(X \le x, Z = 1)$ and $P(X \le x, Z = 0)$ for $x > 0$.
   (**Remark:** You can leave your answer as integrals.)

   (b) Calculate the joint distribution $f_{X,Z}(x, z)$ for $x > 0, z = 0, 1$.