Probability and Statistics Preliminary Examination: May 2026

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); i.i.d. (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I(x \in A)$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbf{E}(X)$: expectation of random variable X.
  - $\mathbf{Var}(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  (a) $Bin(n,p)$: $\mathbf{E}(X) = np$, $\mathbf{Var}(X) = np(1-p)$, and pmf and mgf given respectively by

  $$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} I(x \in \{0, 1, \cdots, n\}), \qquad M(t) = (1 - p + pe^t)^n$$

  (b) $Pois(\lambda)$: $\mathbf{E}(X) = \lambda$, $\mathbf{Var}(X) = \lambda$, and pmf and mgf given respectively by

  $$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  (c) $Geom(p)$: $\mathbf{E}(X) = 1/p$, $\mathbf{Var}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

  $$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1 - (1-p)e^t}, \quad t < -\log(1-p)$$

  (d) $NB(r,p)$: $\mathbf{E}(X) = r(1-p)/p$, $\mathbf{Var}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

  $$f(x) = \binom{r+x-1}{x} p^r (1-p)^x I(x \in \{0, 1, \cdots\}), \qquad M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

  (e) $Gamma(\alpha, \beta)$: $\mathbf{E}(X) = \alpha\beta$, $\mathbf{Var}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

  $$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < 1/\beta$$

  (f) $Exp(\beta)$: Gamma$(1, \beta)$.

(g) $Beta(\alpha,\beta)$: $\mathbf{E}(X)=\frac{\alpha}{\alpha+\beta}$, $\mathbf{Var}(X)=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, and pdf given by

$$f(x)=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}I(0<x<1)$$

(h) Order Statistics: Let $X_{(1)}\le\cdots\le X_{(n)}$ denote the order statistics from a random sample $X_1,\ldots,X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)},X_{(j)})$, and $(X_{(1)},\ldots,X_{(n)})$, are given by:

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j}f(x)I(-\infty<x<\infty)$$

$$f_{X_{(i)},X_{(j)}}(x_i,x_j)=\frac{n!}{(i-1)!(j-1-i)!(n-j)!}f(x_i)f(x_j)[F(x_i)]^{i-1}[F(x_j)-F(x_i)]^{j-1-i}[1-F(x_j)]^{n-j}$$
$$\times\ I(-\infty<x_i\le x_j<\infty)$$

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n)=n!f(x_1)\cdots f(x_n)I(-\infty<x_1\le\cdots\le x_n<\infty)$$

1. Suppose that $X_1, \ldots, X_n$ are i.i.d $N(\theta, 1)$ for $\theta \in \mathbb{R}$. Consider the hypotheses $H_0 : \theta \leq 0$ V.S. $H_1 : \theta > 0$. Find the asymptotic size-α LRT.

2. Suppose that $Y \sim Unif(0, 1)$. Given $Y = y$, the random variable $X \sim \exp(y)$.

   (a) Calculate $\mathbf{P}(X > Y^2)$.

   (b) Calculate $\mathbf{Var}(X)$.

   (**Comment:** The pdf of $\exp(\theta)$ is $\theta^{-1}e^{-x/\theta}$.)

3. Suppose that $X_1, \ldots, X_n$ are i.i.d. from $Pois(\lambda)$ with $\lambda > 0$. Let $\overline{X}$ be the sample mean.

   (a) Find the limiting distribution of $\frac{\sqrt{n}(\overline{X}-\lambda)}{\log(\overline{X})} \xrightarrow{D} ?$

   (b) Use the result in (a) to construct an asymptotic $1 - \alpha$ confidence interval.

4. Let $\theta \in (0, 1)$ be the parameter. Suppose that $X_1, \ldots, X_n$ are i.i.d. observations. For each X_i, it is generated according to the following rule:

$$X_i = \begin{cases} Unif(-0.5, 0.5) & \text{with probability } \theta, \\ Unif(0, 1) & \text{with probability } 1 - \theta. \end{cases}$$

   (a) Find the pdf of X_i.

   (b) Find the MLE of θ.

5. Suppose $X_1, \ldots, X_n$ are i.i.d. $Pois(\lambda)$.

   (a) Calculate $\mathbf{E}\left(\left(1 - \frac{r}{n}\right)^{\sum_{i=1}^n X_i}\right)$, where $0 < r < n$ is a constant.

   (b) Show that $\left(1 - \frac{r}{n}\right)^{\sum_{i=1}^n X_i}$ is the UMVUE of $e^{-r\lambda}$.