Probability and Statistics — Preliminary Examination — 1996

Do 8 of the following 10 problems.

1. Let X be a discrete random variable that can only assume the values $0, 1, 2, \ldots$. Prove that

$$E(X) = \sum_{k=0}^{\infty} P\{X > k\}$$

2. Let $X_1, \ldots, X_n$ be a random sample from the density

$$f(x;\theta) = \begin{cases} \dfrac{\log\theta}{\theta - 1}\theta^x & , \ 0 < x < 1 \ , \ \theta > 1 \\ 0 & , \ \text{elsewhere} \end{cases}$$

(You may find it helpful to realize that $\theta^x = e^{x\log\theta}$.)

(a) Compute $E(X_i), i = 1, \ldots, n$.

(b) Find a sufficient statistic for θ.

(c) Compute the Fisher information in the random sample.

3. Let X and Y be independent random variables with p.d.f.'s $f_x(x) = \begin{cases} \dfrac{1}{\theta}e^{-x/\theta} & , \ x > 0, \theta > 0 \\ 0 & , \ \text{elsewhere} \end{cases}$

and $f_y(y) = \begin{cases} \dfrac{1}{\alpha}e^{-y/\alpha} & , \ y > 0, \alpha > 0 \\ 0 & , \ \text{elsewhere} \end{cases}$, respectively.

(a) Find $P(X \le Y)$

(b) Find $P(X \le t, X \le Y)$

4. Let $X_1, \ldots, X_n$ be i.i.d. random variables with c.d.f. $F(x;\theta)$ determined by the odds ratio

$$\frac{F(x;\theta)}{1 - F(x;\theta)} = R(x;\theta).$$

Find the method of moments estimator of θ if

(a) $R(x;\theta) = e^{(x-\theta)}, x > \theta, -\infty < \theta < \infty$

(b) $R(x;\theta) = (x-\theta)^2, x > \theta, -\infty < \theta < \infty$

$\left(\text{Hint: } \displaystyle\int_0^{\infty} \frac{dt}{(1+t^2)^2} = \frac{\pi}{4} \right)$

5. Let $X_1, \ldots, X_n$ denote a random sample from the uniform distribution on the interval $(0, \theta)$.

(a) Find an unbiased estimator for θ, based on the sample mean $\bar{x}$. Denote it by $\widehat{\theta}_1$.

(b) Find an unbiased estimator for θ, based on the n^{th} order statistic $X_{(n)} = \max(X_1, \ldots, X_n)$. Denote it by $\widehat{\theta}_2$.

(c) Define the efficiency of $\widehat{\theta}_1$ relative to $\widehat{\theta}_2$ by the ratio $\text{Var}(\widehat{\theta}_2) / \text{Var}(\widehat{\theta}_1) = \text{Eff}(\widehat{\theta}_1, \widehat{\theta}_2)$. Find $\text{Eff}(\widehat{\theta}_1, \widehat{\theta}_2)$.

6. Suppose n integers are drawn from $\{1, 2, \ldots, N\}$ at random and with replacement.

(a) Find the method of moments estimatior, $\widehat{N}_1$, of N.

(b) Find $E(\widehat{N}_1)$.

(c) Find Var $(\widehat{N}_1)$.

(d) Find the maximum-likelihood estimator, $\widehat{N}_2$ of N.

7. Suppose that eight observations $X_1, \ldots, X_8$ are drawn at random from a distribution with the following p.d.f.

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & , \ 0 < x < 1 \ , \ \theta > 0 \\ 0 & , \ \text{elsewhere} \end{cases}$$

Show that the UMP test of

$$H_0 : \theta = 1 \text{ against } H_1 : \theta > 1$$

at the α level of significance is given by the rejection region $C = \{(X_1, \ldots, X_8) : \sum_{i=1}^{8} \ln X_i \leq c\}$. Explain how the constant c is found.

8. Let $X_1, \ldots, X_n$ be a random sample from the distribution that has the p.d.f.

$$f(x;\theta) = \begin{cases} \theta^x e^{-\theta}/x! & , \ x = 0, 1, 2, \ldots \ ; \ \theta > 0 \\ 0 & , \ \text{elsewhere} \end{cases}$$

(a) Show that $T = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ.

(b) Construct the best unbiased estimator for θ. Does the variance of this estimator attain the Rao-Cramér bound?

(c) Construct the best unbiased estimator for the parameter θ^2.

9. Let $X_1, \ldots, X_n$ be a random sample of size n from a gamma distribution with p.d.f.

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & , \ x > 0, \ \alpha > 0, \ \beta > 0 \\ 0 & , \ \text{elsewhere} \end{cases}$$

(a) Show that $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\tilde{X} = (\prod_{i=1}^{n} X_i)^{1/n}$ are jointly complete and sufficient for α and β.

(b) For $\alpha = 2$, find the UMVUE of β. Does the variance of this estimator attain the Rao-Cramér bound?

(c) Show that the distribution of $T = \dfrac{\bar{X}}{\tilde{X}}$ does not depend on β. Is T independent of $\bar{X}$? Explain.

10. Let $X_1, \ldots, X_n$ be a random sample from a binomial distribution with p.d.f.

$$f(x;p) = \begin{cases} \binom{m}{x} p^x (1-p)^{m-x} & , \ x = 0, 1, \ldots, m, 0 \leq p \leq 1 \\ 0 & , \ \text{elsewhere} \end{cases}$$

(a) Find the UMVUE of p.

(b) Show that the estimator in part (a) converges in probability to p as $n \to \infty$.