Probability and Statistics
Preliminary Examination
Spring' 97

Please do all eight problems

1. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with mean 0, variance 1, and finite fourth moment. Find the limiting distribution of

$$\frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}}$$

2. Suppose a random sample of size n is taken without replacement from the population $\{1, 2, \cdots, N\}$. Let $Y_{(n)}$ denote the n-th order statistic, i.e. $Y_{(n)}$ is the maximum value in the sample.

   (a) Find the sampling distribution of $Y_{(n)}$, i.e. derive the expression for the p.d.f of $Y_{(n)}$, including its support.

   (b) Find $E(Y_{(n)})$.

   (c) Use part(b) to determine an unbiased estimator for N.

3. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with p.d.f.

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}e^{-\frac{x}{\theta}}, & x > 0, \theta > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Let $\bar{X}$ be the sample mean. Find $E(\bar{X}^2)$.

   (b) Find the unique minimum variance unbiased estimator of var(X). Justify your answer.

   (c) Is the Rao-Cramér lower bound achieved by the variance of the estimator in part(b)?

4. Let X and Y denote the proportions of two different types of components in a sample from a mixture of chemicals used as an insecticide. Suppose X and Y have the joint density function given by

$$f(x,y) = \begin{cases} 2, & 0 \le x \le 1, \ 0 \le y \le 1, \ 0 \le x+y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

   (Note that $X+Y \le 1$ since the random variables denote proportions within the same chemical sample.)

   (a) Find $P(X \le 3/4, \ Y \le 3/4)$.

   (b) Find the marginal density of Y.

   (c) Find the conditional density of X given $Y = y$.

5. Suppose that $X_1, X_2, \cdots, X_n$ is a random sample from a population with p.d.f.

$$f(x;\mu,\sigma) = \begin{cases} \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma}\right)}, & x > \mu, -\infty < \mu < \infty, \ \sigma > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Is this p.d.f. a member of the exponential family of densities?

(b) Find the maximum likelihood estimators of μ and σ^2.

(c) Find the maximum likelihood estimator of $P(X_1 \geq t)$, for $t > \mu$.

6. Let X and Y be independent random variables with c.d.f.'s F and G, respectively. If

$$1 - G(x) = [1 - F(x)]^{\alpha}$$

for some $\alpha > 0$,

(a) find $P(X < Y)$,

(b) if $F(x) = 1 - e^{-x}$, $x > 0$, find the c.d.f. of $Z = \min(X, Y)$.

7. A single observation X is obtained from a Cauchy distribution with p.d.f.

$$f(x;\theta) = \pi^{-1}[1 + (x-\theta)^2]^{-1}, -\infty < x < \infty$$

and it is desired to test $H_0 : \theta = 0$ vs. $H_1 : \theta = 2$.

(a) Find the most powerful (m.p.) test for testing H_0 at level α determined by the rejection region: $\Lambda < 5$ where Λ is the likelihood ratio used for the m.p. test and compute the power of this test for $\theta = 2$.

(b) Consider the alternative test (for H_0) which rejects H_0 if $X > c$ where c is chosen such that this test has the same α as the m.p. test. Compute and compare the powers of these two tests for $\theta = 2$.

8. Let X have one or the other of the discrete distributions given in the following table.

Value of X	1	2	3	5	7
$H_0 : f(x) =$	0.2	0.3	0.1	0.3	0.1
$H_1 : f(x) =$	0.3	0.1	0.3	0.2	0.1

Determine the most powerful (nonrandomized) test for H_0 against H_1 at level $\alpha = 0.3$ and compare the power of this test with that of the test which rejects H_0 if $X = 5$.