Preliminary Exam: Statistics and Probability

May 1998

Work any 8 problems and clearly indicate which problems you wish to be graded.

Begin each problem on a new page, using one side of the sheet. A table of the standard normal distribution is attached.

1. Let p be a given function that is strictly positive and continuous on the real line. Let $P(x) = \int_0^x p(y)dy$, $x \in \mathbb{R}$.

a. For $\theta > 0$ determine the number $c(\theta)$ in such a way that the function

$$f_\theta(x) = \begin{cases} c(\theta)p(x), & 0 \le x \le \theta, \\ 0, & \text{elsewhere,} \end{cases}$$

is a probability density function.

A random sample $X_1, \ldots, X_n$ of size n from the density f_θ is given, where the parameter $\theta > 0$ is unknown.

b. Compute the maximum likelihood estimator for the unknown parameter θ.

c. Find a complete sufficient statistic for θ. Why is it complete?

d. Find the unbiased minimum variance estimator of $P(\theta)$.

e. Determine the conditional expectation $E(P(X_1)|S)$ if S is a complete sufficient statistic for θ.

2. Let X and Y be two independent standard normal random variables and introduce the random variables $U = X + Y$ and $V = X/Y$.

a. Find the joint density of U and V.

b. Find the density of V. What is the name of the density?

c. Are U and V stochastically independent? Justify your answer.

3. Let $X_1, \ldots, X_n$ be a random sample of size n from the probability density function

$$f_\theta(x) = \begin{cases} \theta^2 x e^{-\theta x}, & 0 \le x < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

with the parameter $\theta > 0$ unknown.

a. Suppose we have a sample of size $n = 1$. In testing the null hypothesis $H_0 : \theta = 1$ versus the alternative $H_1 : \theta > 1$, let the null hypothesis be rejected if and only if $X_1 \leq 1$. Find the power function and size of this test.

In part **b.** and **c.** let the sample size n be arbitrary.

b. Find the family of most powerful tests for testing the null hypothesis $H_0 : \theta = 1$ versus the alternative $H_1 : \theta = 2$.

c. Determine the family of uniformly most powerful tests for testing the null hypothesis $H_0 : \theta \leq 1$ versus the alternative $H_1 : \theta > 1$.

4. Let $X_1, \ldots, X_{100}$ be a random sample of size $n = 100$ from the gamma distribution having density

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \; x \geq 0,$$

with $\alpha = 5$ and $\beta = 3$.

a. Find the moment generating function of $Y = \sum_{i=1}^{100} X_i$.

b. What is the name of the distribution of Y?

c. Find the moment generating function of $\overline{X} = Y/100$.

d. What is the name of the distribution of $\overline{X}$?

e. Using the central limit theorem, approximate the probability that $\overline{X}$ is at most 14.

5. Let $X_1, \ldots, X_n$ be a random sample of size n from a distribution with probability density function

$$f_\theta(x) = \begin{cases} 1/\theta, & 0 \leq x \leq \theta, \\ 0, & \text{elsewhere,} \end{cases}$$

for some unknown $\theta > 0$ and let $Y_n = \max\{X_1, \ldots, X_n\}$. Suppose the null hypothesis $H_0 : \theta = 1$ is rejected in favor of the alternative $H_1 : \theta > 1$ if and only if $Y_n \geq c$.

a. Find the number c such that the significance level of the test is $\alpha = .05$.

b. Determine the power function of this test.

6. Let $X_1, \ldots, X_n$ be a random sample from a discrete distribution with probability density function

$$f_\theta(x) = \begin{cases} (1-\theta)\theta^x, & x = 0, 1, 2, \ldots \\ 0, & \text{elsewhere,} \end{cases}$$

with $0 < \theta < 1$ unknown.

a. Show that $E(X_1) = \theta/(1-\theta)$ and $\text{Var}(X_1) = \theta/(1-\theta)^2$.

b. Show that $\sum_{i=1}^n X_i$ is a sufficient and complete statistic for θ.

c. Find the Cramér-Rao lower bound for an unbiased estimator of θ.

d. Find the unbiased minimum variance estimator of $\theta/(1-\theta)$.

7. Let X_1, X_2, X_3 be a random sample from a distribution with probability density function f which is strictly positive and continuous on the entire real line. Let $Y_1 = \min\{X_1, X_2, X_3\}$ and $Y_3 = \max\{X_1, X_2, X_3\}$ and define the random variable

$$W = \int_{Y_1}^{Y_3} f(x)dx.$$

a. Prove that the distribution of W does not depend on the density f when it satisfies the conditions mentioned above.

b. Compute the probability density function of the random variable W.

8. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with unknown mean $\theta \in \mathbb{R}$ and variance 1. We want to test the null hypothesis $H_0 : \theta = 0$ versus the alternative $H_1 : \theta < 0$. As an alternative to the uniformly most powerful test one may use the sign test. To describe this test let $Y_i = 1$ if $X_i < 0$ and $Y_i = 0$ if $X_i \geq 0$. The sign test rejects H_0 if and only if $\sum_{i=1}^n Y_i \geq d$, for a suitable number $d \geq 0$, depending on the size of the test.

a. What is the exact distribution of $\sum_{i=1}^n Y_i$ under H_0?

b. Use the central limit theorem to determine the number d in the description of the sign test in such a way that this test has approximate size $\alpha \in (0, 1)$.

c. Using the central limit theorem, approximate the power of the sign test derived under **b.** at the alternative $\theta = -\frac{1}{2}$.

9. Let X and Y be stochastically independent random variables. Suppose that X has a normal distribution with mean 1 and variance 2, and that Y has a normal distribution with mean 2 and variance 1.

a. Find the number $a \in \mathbb{R}$ such that $aX + Y$ and $(X - Y)^2$ are stochastically independent.

(**Hint:** you may use the fact that two normally distributed random variables are stochastically independent if their covariance equals 0.)

b. Find the expected value of $(X + Y)^2$.