Preliminary Exam: Statistics and Probability
May, 1999

Instructions: Work on one side of the sheet only. Start each problem on a new page. Be complete and concise on each problem.

1. Let $X_1, ..., X_m$ and $Y_1, ..., Y_n$ be two independent random samples from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ populations, respectively. Find the maximum likelihood estimate of $\theta = (\mu_1, \mu_2, \sigma^2)$.

2. Let the random variable X have a binomial distribution with the pdf

$$f(x;\theta) = \begin{cases} \binom{n}{x} \theta^x (1-\theta)^{n-x}, & x = 0, 1, ..., n, \ 0 < \theta < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the conditional probability

$$P(X = x | X \geq 1) \text{ for any } x = 1, ..., n.$$

(b) Let Y have the pdf

$$f(y;\theta) = \begin{cases} \frac{\binom{n}{y} \theta^y (1-\theta)^{n-y}}{1-(1-\theta)^n}, & y = 1, ..., n, \ 0 < \theta < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(i) Show that Y is a complete and sufficient statistic for θ.

(ii) Show that $\frac{Y}{n}$ is the best estimator of $\frac{\theta}{1-(1-\theta)^n}$.

3. Construct the likelihood ratio test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ using data from a random sample drawn from a population with the pdf

$$f(x) = \begin{cases} \theta e^{-\theta x}, & x > 0, \theta > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that the critical region is of the type $\bar{X} \exp(-\theta_0 \bar{X}) >$ constant.

(b) Sketch the graph of this function of $\bar{X}$ and indicate the corresponding critical region in terms of $\bar{X}$.

(c) Find an expression for the power of this test for $\theta = \theta_1 > \theta_0$.

4. In adding 100 real numbers, each is rounded off to the nearest integer. Assume that the round-off errors of these 100 observations are independent and have the same uniform distribution over the interval (- 0.5, 0.5). Determine the approximate probability that the error in the sum of these 100 numbers is no greater than 5 in magnitude.

5. Suppose the length of time, X, it takes a worker to complete a certain task has the pdf

$$f(x) = \begin{cases} e^{-(x-\theta)}, & x > \theta \\ 0, & \text{elsewhere} \end{cases}$$

where θ is a positive constant that represents the minimum time to task completion. Let $X_1, ..., X_n$ denote a random sample of completion times from this distribution.

a. Find the pdf of $Y_{(1)} = \min(X_1, ..., X_n)$.

b. Find $E[Y_{(1)}]$.

c. Find the maximum likelihood estimator for θ.

d. Find a minimum-variance-unbiased estimator for θ.

6. Three prisoners are informed by their jailer that one of them has been chosen at random to be executed, and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information, since he already knows that at least one will go free. The jailer refuses to answer this question, pointing out that if A knew which of his fellows were to be set free, then his own probability of being executed would rise from 1/3 to 1/2, since he would then be one of two prisoners. What do you think of the jailer's reasoning?

7. Let X, Y, and Z be independent random variables each having the uniform distribution on the interval $[0, 1]$.

a. Find the density of XY.

b. Find the density of Z^2.

c. Find the joint density of XY and Z^2.

d. Compute $P\{XY < Z^2\}$.

8. Let $X_1, ..., X_m$ be a random sample from the pdf

$$f_{\theta_1}(x) = \begin{cases} \theta_1 x^{\theta_1 - 1}, & 0 < x < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

for some $\theta_1 > 0$. Set $U_i = -\log X_i$, $i = 1, ..., m$.

a. Compute the pdf of the random variable $W_i = 2\theta_1 U_i$. What is the name of the corresponding distribution?

b. What is the name of the distribution of $2\theta_1 \sum_{i=1}^{m} U_i$?

Next let $Y_1, ..., Y_n$ be a random sample from the pdf

$$f_{\theta_2}(y) = \begin{cases} \theta_2 y^{\theta_2 - 1}, & 0 < y < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

for some $\theta_2 > 0$. This sample is independent of the first one. Set $V_j = -\log Y_j$, $j = 1, ..., n$.

c. Find the likelihood ratio test for testing the null hypothesis $H_0 : \theta_1 = \theta_2$ versus the alternative $H_1 : \theta_1 \neq \theta_2$.

d. Show that the likelihood ratio test can be expressed in terms of the statistic

$$T = \frac{\sum_{i=1}^m U_i}{\sum_{i=1}^m U_i + \sum_{j=1}^n V_j}.$$

e. What is the name of the distribution of the statistic $\{(m+n)/n\}T$ under the null hypothesis? (Hint: use part **b.**)

9. Let $X_1, ..., X_n$ be a random sample from the pdf

$$f_\theta(x) = \begin{cases} \theta(1+x)^{-(1+\theta)}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

for some $\theta > 0$.

a. Find a complete and sufficient statistic and motivate your answer.

b. Find the maximum likelihood estimator of $1/\theta$.

c. Find the Cramér-Rao lower bound for unbiased estimators of $1/\theta$.

d. Find the uniform minimum variance unbiased estimator of $1/\theta$.

10. Let X and C be independent random variables having c.d.f.'s $F(t)$ and $G(t)$, respectively.

Define $\delta = 1$ if $X \leq C$ and $\delta = 0$ otherwise.

a. Find the conditional c.d.f. $P[X \leq t | \delta = 1]$, $t \in R$ and the conditional c.d.f. $P[X \leq t | \delta = 0]$, $t \in R$.

b. First obtain $P[\delta = 1 | C \in (t, t + \Delta t)]$, then take the limit as $\Delta t \downarrow 0$ to get the conditional probability $P[\delta = 1 | C = t]$.