Preliminary Exam: Statistics and Probability

May, 2000

Instructions: Work on one side of the page only. Start each problem on a new page. Be complete and concise on each problem. Please turn in the solutions to exactly 8 problems.

Work any 3 of the following 5 problems. Turn in only 3 solutions.

1. A random variable, X, with cumulative distribution function given by

$$F(x) = [1 - \exp(-x^2)]^\theta,\ x > 0,\ \theta > 0$$

is said to be a *Burr Type X* random variable.

Let $X_1, X_2, ..., X_n$ be a random sample from a *Burr Type X* population with parameter θ.

- **(a)** Show that the *Burr Type X* is in the exponential class of distributions.
- **(b)** Find the maximum likelihood estimate of $\theta, \hat{\theta}$.
- **(c)** Show that $\hat{\theta}$ is a complete and sufficient statistic for θ.
- **(d)** Find $E[1/\hat{\theta}]$.

2. Let $X_1, ..., X_n$ be a random sample of size n from the Poisson distribution with parameter $\theta > 0$, and let $r \in \mathbb{N}$ be a given integer.

- **(a)** Determine a complete and sufficient statistic for θ.
- **(b)** Find the uniform minimum variance estimator of θ^r.

3. Define the following terms for the problem of estimating $g(\theta)$ with a loss function $L(\theta, d)$.

- **(a)** An UMVU estimator;
- **(b)** The risk function of an estimator δ;
- **(c)** A minimax estimator;
- **(d)** An admissible estimator;
- **(e)** A Bayes estimator with respect to a prior distribution λ.

4. Let $X_1, ..., X_n$ be i.i.d. random variables from a distribution whose density function is

$$f(x, \theta) = \begin{cases} \frac{1}{\theta}, & \text{if} \quad 0 < x < \theta \\ 0, & \quad \text{otherwise,} \end{cases}$$

for some $\theta > 0$.

(a) Formally compute the Cramér-Rao lower bound

$$\frac{1}{nE\left[\left(\frac{\partial \log f}{\partial \theta}\right)^2\right]}.$$

(b) Show that $\hat{\theta} = \frac{n+1}{n} X_{(n)}$ is the MVUE of θ where $X_{(n)}$ is the maximum of $X_1, ..., X_n$.

(c) Compute the variance of $\hat{\theta}$ and compare it with the Cramér-Rao lower bound.

5. Suppose that f is the pdf of a continuous real-valued random variable, such that the following regularity conditions are satisfied:

(i) $f > 0$ on $\mathbb{R}$, f' exists and is continuous on $\mathbb{R}$;

(ii) $\lim_{|x| \to \infty} x f(x) = 0$;

(iii) $\int_{-\infty}^{\infty} [x^2 \{f'(x)\}^2 / f(x)] dx < \infty$.

(a) Show that for each $\theta > 0$ the function $f_\theta(x) = (1/\theta) f(x/\theta)$, $x \in \mathbb{R}$, is also a probability density function.

(b) Prove that $E\left[\frac{d}{d\theta} \log f_\theta(X)\right] = 0$.

(c) Compute the Fisher information about θ contained in X.

(d) What is the Cramér-Rao lower bound for estimating θ^2, given a random sample of size n from f_θ?

Work any 3 of the following 4 problems. Turn in only 3 solutions.

6. Let X be a random sample of size 1 from a probability distribution P on the real line. Let Φ denote the cumulative distribution function of the standard normal distribution. Determine the most powerful test of level $2(1 - \Phi(1))$ for testing the null hypothesis that P has the standard normal density against the alternative that P has density $(1/4)\, e^{-|x|/2}$, $x \in \mathbb{R}$.

7. Let $X_1, ..., X_{10}$ be a random sample of size 10 from a Poisson distribution with mean θ.

(a) Show that the critical region C defined by $\sum_{i=1}^{10} x_i \geq 3$ is a best critical region for testing $H_0 : \theta = 0.1$ against $H_1 : \theta = 0.5$.

(b) Determine the significance level α, and the power of the test at $\theta = 0.5$.

8. Let $X_1, ..., X_n$ be a random sample of size n from a distribution with pdf $f(x)$ that is symmetric about 0, i.e. $f(x) = f(-x)$, for all x. Define $T =$ [number of X_i's for which $|X_i| > 1$].

(a) Show that T has a binomial distribution with parameters n and p, where $p = P\{|X| > 1\}$

(b) Assume that we wish to test $H_0 : p = 1/2$ against $H_1 : p > 1/2$ based on a random sample of size $n = 10$. Find the significance level of the test if the critical region is given by $T \geq 8$.

9. Let X and Y be i.i.d. continuous, uniform (0,1) random variables. Find the distribution of $T = X - Y$.

Work any 2 of the following 3 problems. Turn in only 2 solutions.

10. Let Y be a $\Gamma(\alpha, \beta)$ r.v. with $\alpha = \frac{1}{\beta}$ $(\beta > 0)$. Given $Y = y$, the conditional pdf of X is $e^{-y\Lambda(x)} y\lambda(x)$, where $\Lambda(x)$ is a smooth non-negative function, with $\lim_{x \to -\infty} \Lambda(x) = 0$, $\lim_{x \to -\infty} \Lambda(x) = \infty$, and $\lambda(x) = \frac{d}{dx}\Lambda(x)$.

(a) Find the joint pdf of (X, Y).

(b) From (a) obtain the (marginal) pdf of X.

(c) If $\Lambda(x) = \begin{cases} \gamma x & x \geq 0 \\ 0 & x < 0 \end{cases}$, $\gamma > 0$, obtain the equations that the MLE of (γ, β) must satisfy, based on a random sample $X_1, ..., X_n$ from the pdf of X in (b).

11. Let X and Y be independent, with cdf's F and G, respectively. Assume $1 - G = (1 - F)^\beta$ for some $\beta > 0$ and that both F and G have a density.

(a) Express $P(X < Y)$ in terms of β.

(b) Find the c.d.f. of $\min(X, Y)$.

12. Let X be a random variable with pdf

$$f(x|P) = \begin{cases} P, & x = 0 \\ (1 - P)g(x), & x \in \mathcal{A} \backslash 0 \end{cases},$$

where P is between 0 and 1, and $g(x)$ is a function not dependent on P such that $\int_{\mathcal{A}} g(x) = 1$. Further assume that the prior distribution of P is beta with parameters α and β. That is,

$$\pi(p|\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha - 1}(1 - p)^{\beta - 1}, \ p \in [0, 1]$$

is the prior pdf for P.

(a) Find the posterior distribution, that is, the conditional distribution of P, given $X = x$.

(b) Find the Bayes estimate for P under squared-error loss.