Preliminary Exam: Probability and Statistics

May, 2001

Instructions: Work on one side of the page only. Start each problem on a new page. Be complete and concise on each problem.

Do exactly 2 of problems 1-3, exactly 2 of problems 4-6, and all of the remaining problems.

1. The lognormal (α, β) distribution is the distribution of the random variable $Y = e^X$, where X has a $N(\alpha, \beta^2)$ distribution. Derive the mean and variance of this lognormal distribution using the gamma function

$$\Gamma(\theta) = \int_0^\infty x^{\theta-1} e^{-x} dx, \ \theta > 0.$$

2. Let the random variables X and Y have the joint p.d.f.

$$f_{X,Y}(x,y) = \begin{cases} \frac{x^3}{2} e^{-x(1+y)} & , \ y > 0, \ x > 0 \\ 0 & , \text{ elsewhere.} \end{cases}$$

(a) Determine the marginal p.d.f. f_X of X.

(b) Compute the conditional expectation $E(Y|X = x)$.

(c) Are the random variables X and Y independent? Explain.

3. Let X be a random variable with moment generating function $M(t)$ which exists and is finite for $t \in (c, d)$, where c, d are specified real numbers $(c < 0 < d)$.

(a) Show that if a is any real number, then $P(X \geq a) \leq M(t)e^{(-at)}$, for every $t \in (0, d)$.

(b) Since the left-hand side of the above inequality does not depend on $t \in (0, d)$ this inequality remains true even if we minimize the right-hand side of the inequality over the interval $(0, d)$.

Compute this minimum for the particular case when $X \sim N(0, 1)$ and compare this minimum upper bound with the exact value of $P(X \geq a)$ when $a = 3$.

4. Let X_1 and X_2 be a random sample of size 2 from a distribution with p.d.f.

$$f(x;\theta) = \begin{cases} \theta x^{\theta - 1} & , 0 < x < 1, \ \theta > 0 \\ 0 & , \text{ elsewhere.} \end{cases}$$

(a) Find the cumulative distribution function of the random variable $Y = X_1X_2$ for arbitrary $\theta > 0$.

(b) Consider testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$. Using this sample of size 2, find the best critical region of size $\alpha = 1 - \frac{2}{e}$.

5. Let $X_1, ..., X_n$ be a random sample of size n from a distribution with p.d.f.

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}x^{(1-\theta)/\theta} & , 0 < x < 1,\ \theta > 0 \\ 0 & , \text{elsewhere.} \end{cases}$$

(a) Find a uniformly most powerful test of size $0 < \alpha < 1$ for testing

$$H_0 : \theta \leq \theta_0 \text{ against } H_1 : \theta > \theta_0, \text{ where } \theta_0 > 0 \text{ is known.}$$

(b) For $n = 2, \theta_0 = 1$, and $\alpha = 0.05$, find the power function of the test in part (a).

6. Let X_1 and X_2 be independent random variables. Suppose that X_i has a uniform $(0, \theta_i)$ distribution, with $\theta_i > 0$, $i = 1, 2$. Consider the test that rejects $H_0 : \theta_1 = \theta_2$ in favor of $H_1 : \theta_1 > \theta_2$ when $X_1 > cX_2$ for some $c > 0$.

(a) Find the power function for this test.

(b) What must c be if the test is to be run at the $\alpha = 0.05$ level of significance?

7. Let $Y_1, ..., Y_n$ be independent random variables. Assume that Y_i has a $N(c_i\mu, \sigma^2)$ distribution, $i = 1, ..., n$, where $c_1, ..., c_n$ are known constants and μ and $\sigma^2 > 0$ are unknown real numbers.

(a) Find the maximum likelihood estimators of μ and σ^2.

(b) Find the likelihood ratio test of size α for testing

$$H_0 : \mu = 0 \text{ against } H_1 : \mu > 0,$$

and show that the test statistic can be expressed in terms of a statistic having a t-distribution.

8. Let $X_1, ..., X_n$ denote a random sample from the Poisson distribution

$$f(x;\lambda) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!}, & x = 0, 1, 2, \cdots,\ \lambda > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

Define $U(x)$ by

$$U(x) = \begin{cases} 1, & \text{if } x_1 = 0 \\ 0, & \text{otherwise} \end{cases}$$

Determine the unique best estimator for the parameter $e^{-\lambda}$ by completing the following steps.

(a) Find $E(U(X_1))$.

(b) Find $P(X_1 = 0 | \sum_{i=1}^{n} X_i = y)$, explicitly as a function of y.

(c) Find $\varphi(y) = E(U(X_1) | \sum_{i=1}^{n} X_i = y)$.

(d) Using the result in (c) state the unique best estimator for $e^{-\lambda}$.

9. Let $X_1, ..., X_n$ denote a random sample from the Poisson distribution

$$f(x; \lambda) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!}, & x = 0, 1, 2, \cdots, \ \lambda > 0, \\ 0 & , \text{elsewhere} \end{cases}$$

(a) Find the unique best estimator of λ^2.

(b) Find the Cramer-Rao lower bound for the variance of the estimator in part (a).