Preliminary Exam: Probability and Statistics
May, 2002

Instructions: Work on one side of the page only. Start each problem on a new page. Be complete and concise on each problem. Do exactly 2 problems from group I, exactly 3 problems from group II and exactly 3 problems from group III. Please turn in the solutions to exactly 8 problems.

Group I

1. Let X have a standard normal distribution. Observe X, then toss a fair coin and define Y as follows:

$$Y = \begin{cases} X & \text{if the coin falls "Heads"} \\ -X & \text{if the coin falls "Tails"} \end{cases}$$

   a. Show that Y is normally distrubuted.

   b. Find the mean and variance of $S = X + Y$

   c. Is S also normally distributed? Justify your answer.

2. Consider the function

$$f(x,y) = \begin{cases} 4xy - 2x - 2y + 2 & , 0 < x, y < 1 \\ 0 & , \text{elsewhere,} \end{cases}$$

   a. Show that f is a bivariate density function.
   Let X and Y have f as their joint density function. Determine.

   b. the distribution of X;

   c. the conditional distribution of Y, given that $X = x$, for $0 < x < 1$;

   d. the mean of this conditional distribution;

   e. the variance of this conditional distribution.

3. $Y_1, Y_2 \sim N(\theta, 1)$. Consider $\hat{\theta}^* = E(Y_1|\overline{Y})$
   Find

   a. Find $f(y_1|\overline{y})$.

   b. Why doesn't $f(y_1|\overline{y})$ depend on θ?

   c. Find $\hat{\theta}^*$.

   d. Find $E\left[\hat{\theta}^*\right]$ and $V\left[\hat{\theta}^*\right]$.

Group II

1. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the normal distribution $N(\mu, \sigma^2)$. Consider the following integral

$$\frac{1}{\sqrt{2\pi}\sigma} \int_A^\infty e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx = .05.$$

   Find the UMVUE for A.

2. Let two independent random samples of sizes $n = 16$ and $m = 10$, taken from two independent normal distributions $n(\mu_1, \sigma_1^2)$ and $n(\mu_2, \sigma_2^2)$ respectively, yield $\overline{x} = 3.6$, $s^2, = 4.14$, $\overline{y} = 13.6$, $s_2^2 = 7.26$.

   a. Find a 90% confidence interval for σ_1^2/σ_2^2 when μ_1, and μ_2 are unknown.

   b. Do you use the same confidence interval if it is known that $\mu_1 = 4.27$ and $\mu_2 = 14.72$? Explain with appropriate details.

3. Let $X_1, \ldots, X_n$ be independent, positive and continuous r.v.'s having p.d.f. f_i and c.d.f. F_i, $i = 1, \ldots, n$, respectively.

$$\text{Define} \quad \lambda_i(t) = \frac{f_i(t)}{1 - F_i(t)}.$$

   a. Show that $F_i(t) = 1 - e^{-\int_0^t \lambda_i(s)ds}$

   b. Suppose $\lambda_i(t) = \lambda_0(t)e^{c_i\beta}$, where λ_0 is a known function, $c_i's$ known constants, and β the unknown parameter. Obtain the equation the $MLE\ \widehat{\beta}$ must satisfy.

   c. If $c_i = \begin{cases} 1 & i \leq n_1 \\ 0 & \text{otherwise,} \end{cases}$ $n_1 < n$ and $\lambda_0(t) = 1$, simplify and work out (b)

4. Consider the family of exponential densities

$$f(x, \theta) = \begin{cases} \frac{1}{\theta} \quad e^{-x/\theta} & , x \geq 0, \\ 0 & , \text{elsewhere,} \end{cases} \qquad \theta > 0.$$

   You may use that

$$\int_0^\infty x^n f(x, \theta)\, dx = n!\, \theta^n.$$

   a. Let X have density $f(x, \theta)$, for some $\theta > 0$. Determine the Fisher information $I(\theta)$ based on this single observation.

   Let X_1, X_2 be a random sample of size $n = 2$ from the density $f(x, \theta)$, for some $\theta > 0$, and let $S^2 = \frac{1}{n-1}\sum_{i=1}^{2}(X_i - \overline{X})^2 = (X_1 - \overline{X})^2 + (X_2 - \overline{X})^2$ be the sample variance ($\overline{X} = \frac{1}{2}(X_1 + X_2)$).

   b. Verify that S^2 is an unbiased estimator of $\tau(\theta) = \theta^2$.

   c. Compute $Var_\theta S^2$ and the Cramér - Rao lower bound for estimating $\tau(\theta) = \theta^2$ based on a sample of size $n = 2$.

   d. Is S^2 an efficient estimaor of θ^2? (Motivate your answer.)
   (Hint: the Cremér -Rao Lower bound for the variance of unbiased estimators of $\tau(\theta)$ is given by $\frac{(\tau'(\theta))^2}{nI(\theta)}$ where $I(\theta)$ is the Fisher information.)

Group III

1. Let $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$ be random samples from normal distributions $n(\mu_1, 400)$ and $n(\mu_2, 225)$ respectively. Construct a test for $H_0 : \theta = 0$ against $H, : \theta > 0$ where $\theta = \mu_1 - \mu_2$, based on the difference between the two sample means.

   a. Choose your critical region (including the choice of n) such that if $K(\theta)$ denotes the power function of the test then $K(0) = .05$ and $K(10) = 0.90$.

   b. Is this test unbiased in the sense that $K(\theta) \geqslant 0.05$ for all $\theta > 0$? Explain with appropriate details.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a continuous Uniform$(\theta - 1/2, \theta + 1/2)$ population. Consider the two-sided hypothesis test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

   a Find the likelihood ratio test statistic as a function of $min(X_1, X_2, \ldots, X_n)$ and $max(X_1, X_2, \ldots, X_n)$.

   b Find the level of significance for the likelihood ratio test.

3. Let y be a single observation from the p.d.f. $f(y; \sigma) = \sigma e^{-\sigma y}, y \geq 0$. Determine the likelihood ratio critical region for testing $H_0 : \sigma = 1$ against $H_1 : \sigma \neq 1$.

4. Assume that $Y_1, Y_2, \ldots, Y_n$ denotes a random sample of size n from a normal distribution with unknown mean μ and unknown variance σ^2. Consider testing of the following hypothesis:

$$\begin{aligned} H_0 &: \quad \mu = 0 \\ H_a &: \quad \mu \neq 0 \end{aligned}$$

   Show that the likelihood ratio test statistic is equivalent to the t-test statisitic.