Preliminary Exam: Probability and Statistics
August 2003

Work all 8 problems. Begin each problem on a new page, using one side of the sheet. Throughout "p.d.f." means "probability density function", $\mathbb{R}$ is the real line $(-\infty, \infty)$, and $\mathbb{N}$ the set of natural numbers $\{1, 2, \dots\}$. A table of the standard normal distribution is attached.

1. A certain manufacturing process produces vacuum tubes whose lifetimes in hours are independent random variables with the Exponential distribution with mean 1,500 hours. Find an approximation of the probability that the total life of 50 tubes will exceed 80,000 hours.

2. A test correctly identifies a disease D with probability 0.95 and wrongly diagnoses D with probability 0.01. From past experience, it is known that disease D occurs in a targeted population with frequency 0.2%. An individual is chosen at random from said population and is given the test. Calculate the probability that

- **a.** the test is positive: $P(+)$;
- **b.** the individual actually suffers from disease D if the test turns out to be positive: $P(D|+)$.

3. Let the discrete random variables X and Y have the joint p.d.f.

$$f_{X,Y}(x, y) = \frac{2}{n(n+1)}, \ y = 1, \dots, x, \quad x = 1, \dots, n.$$

Then compute

- **a.** the marginal p.d.f.'s f_X and f_Y;
- **b.** the conditional p.d.f.'s $f_{X|Y}(\bullet|y)$ and $f_{Y|X}(\bullet|\, x)$;
- **c.** the conditional expectations $E(X|y)$ and $E(Y|x)$.

(Hint: Recall that $\sum_{k=1}^{m} k = \frac{1}{2}m(m+1)$ for each $m \in \mathbb{N}$.)

4. Suppose a random sample of size 2 from the Cauchy p.d.f.

$$f_\theta(x) = \frac{1}{\pi\{1 + (x-\theta)^2\}}, x \in \mathbb{R}, \theta \in \mathbb{R},$$

results in the values $x_1 = -1$ and $x_2 = 1$. For these values of the sample elements compute the value of the maximum likelihood estimator of θ.

5. Let $X_1, \dots, X_n$ be a random sample of size n from the Poisson distribution with parameter $0 < \theta < \infty$.

- **a.** Compute the Fisher information $I(\theta)$ given this family of distributions.
- **b.** Construct a confidence interval for θ with confidence coefficient approximately equal to $1 - \alpha$, for some $0 < \alpha < 1$. (Assume n is sufficiently large.)

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6. The number of times that an electric light switch can be turned on and off until failure occurs is a random variable which may be assumed to have the Geometric p.d.f. with parameter θ, i.e. the p.d.f.

$$f_\theta(x) = \theta(1-\theta)^x, x = 0, 1, \ldots.$$

- **a.** Show that these densities form an exponential family.
- **b.** Given a random sample $X_1, \ldots, X_n$ of size n from this p.d.f. f_θ, derive the Uniformly Most Powerful test for testing the null hypothesis $H_0 : \theta = \frac{1}{2}$ against the alternative $H_1 : \theta > \frac{1}{2}$ at the level of significance $\alpha = 0.05$.
- **c.** Determine an interval around 0 where the moment generating function of a random variable with p.d.f. f_θ exists. Compute the moment generating function explicitly on that interval.
- **d.** Find the mean μ_θ and variance σ_θ^2 of a random variable with p.d.f. f_θ.
- **e.** Use the Central Limit Theorem to find an approximate value for the cut-off point of the test in **b**.

7. Let X be a random variable with the Double Exponential p.d.f.

$$f_\theta(x) = \frac{1}{2\theta}\, e^{-|x|/\theta}, x \in \mathbb{R}, 0 < \theta < \infty,$$

and suppose $X_1, \ldots, X_n$ is a random sample from this distribution.

- **a.** Compute $E_\theta|X|$ and $E_\theta X^2$.
- **b.** Show that the statistic $T = \frac{1}{n}\sum_{i=1}^n |X_i|$ is an unbiased estimator of θ, and calculate its variance.
- **c.** Calculate the Fisher information $I(\theta)$ given the above family of p.d.f.'s.
- **d.** Is T in part **b** the Uniform Minimum Variance Unbiased estimator of θ? Why?

8. Let $X_1, \ldots, X_n$ be a random sample of size n from the Exponential p.d.f.

$$f_\theta(x) = \begin{cases} \theta e^{-\theta x}, 0 < x < \infty, \\ 0, \qquad \text{elsewhere}, \end{cases} \tag{1}$$

where $0 < \theta < \infty$. The null hypothesis $H_0 : \theta = 1$ is to be tested against the alternative $H_1 : \theta \neq 1$.

- **a.** Find the Likelihood Ratio statistic λ for this testing problem, and indicate the critical regions by graphing.
- **b.** What is the limiting distribution of $-2\log\lambda$ under H_0? (Here "log" is the natural logarithm.)