Preliminary Exam: Probability and Statistics
August 2004

Work all 8 problems. Begin each problem on a new page using one side of the sheet. Throughout "p.d.f." means "probability density function", $\mathbb{N} = \{1, 2, \dots\}$, and $\mathbb{R} = (-\infty, \infty)$. A table of the standard normal distribution is attached.

1. Let the random variables X and Y have joint p.d.f.

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{2}\, x^3 e^{-x(1+y)}, & x > 0, y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Determine the marginal p.d.f. f_X of X.

b. Compute the conditional expectation of Y given $X = x$.

c. Are X and Y independent? Explain.

2. In adding 100 real numbers, each is rounded off to the nearest integer. Assume that the round-off errors of these 100 observations are stochastically independent and have the uniform distribution over the interval $(-0.5, 0.5)$. Find an approximation of the probability that the error in the sum of these 100 rounded off numbers is not greater than 5.

3. Let X be a sample of size 1 from the p.d.f.

$$f_\theta(x) = \begin{cases} \dfrac{1}{\theta}, & x = 1, \dots, \theta, \\ 0, & \text{elsewhere,} \end{cases}$$

for some $\theta \in \mathbb{N}$.

a. Show that the statistic X is complete for $\theta \in \mathbb{N}$.

b. Show that $2X - 1$ is the uniform minimum variance unbiased estimator of θ.

4. Let $X_1, \dots, X_n$ be independent random variables. Assume that X_i has a normal distribution with mean $c_i\mu$ and variance σ^2, where $c_1, \dots, c_n$ are known numbers that are not all 0, and $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ are unknown parameters.

a. Find the maximum likelihood estimators of μ and σ^2.

b. Find the family of likelihood ratio tests for testing $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$.

5. Suppose that $X_1, \dots, X_n$ is a random sample from the p.d.f.

$$f_\theta(x) = \begin{cases} \theta(1 + x)^{-(1+\theta)}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

for some $\theta > 0$.

a. Find a complete and sufficient statistic for $\theta > 0$.

b. Find the maximum likelihood estimator of $1/\theta$.

c. Find the Cramér-Rao lower bound for unbiased estimators of $1/\theta$.

d. Find the uniform minimum variance unbiased estimator of $1/\theta$.

e. Is the estimator in part **d** efficient for estimating $1/\theta$?

6. Suppose that $X_1, \ldots, X_n$ is a random sample of size n from the p.d.f.

$$f_\theta(x) = \begin{cases} \dfrac{1}{\theta}, & 0 < x < \theta, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\theta > 0$, and let $Y = \max\{X_1, \ldots, X_n\}$. Let the null hypothesis $H_0 : \theta = 1$ be rejected in favor of the alternative $H_1 : \theta > 1$ if and only if $Y \geq c$, for some $c \in \mathbb{R}$.

a. Find the number c so that the significance level of the test is $\alpha = 0.05$.

b. Determine the power function of the test with c as in part **a**.

7. Let X and Y be stochastically independent random variables. Suppose that X has a normal distribution with mean 1 and variance 2, and that Y has a normal distribution with mean 2 and variance 1.

a. Find the number $a \in \mathbb{R}$ such that $aX + Y$ and $(X - Y)^2$ are stochastically independent. (**Hint**: you may use the fact that two normally distributed random variables are stochastically independent if their covariance equals 0.)

b. Find $E(X + Y)^2$.

8. Let X be a random sample of size 1 from a probability distribution P on the real line. Let Φ denote the cumulative distribution function of the standard normal distribution. Determine the most powerful test of level $2\{1 - \Phi(1)\}$ for testing the null hypothesis that P has the standard normal p.d.f. $\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$, $x \in \mathbb{R}$, against the alternative that P has p.d.f. $\psi(x) = \frac{1}{4}\, e^{-|x|/2}$, $x \in \mathbb{R}$.