Preliminary Exam: Probability and Statistics
May 2005

Work all 8 problems. Begin each problem on a new page, using one side of the sheet. Throughout "p.d.f." means "probability density function", $\mathbb{R} = (-\infty, \infty)$, and $\mathbf{1}_A$ is the indicator function of the set A (i.e. $\mathbf{1}_A(x) = 1$ if $x \in A, 0$ otherwise). A table of the standard normal distribution is attached.

1. Suppose that we are given a random sample $X_1, \dots, X_n$ from the p.d.f.

$$f_\theta(x) = \theta x^{\theta-1}\mathbf{1}_{(0,1)}(x), \quad x \in \mathbb{R},$$

where $\theta > 0$ is an unknown parameter. The null hypothesis $H_0 : \theta = 1$ is to be tested against the alternative $H_1 : \theta > 1$.

- **a.** Determine the family of uniformly most powerful tests.
- **b.** Assuming that the sample size n is sufficiently large, use the central limit theorem to find a uniformly most powerful test of approximate significance level $\alpha = 0.05$.
- **c.** For sample size $n = 1$ find the uniformly most powerful test of exact significance level $\alpha = 0.05$.

2. Let $X_1, \dots, X_n$ be a random sample of size n from the p.d.f.

$$f_\theta(x) = \frac{1}{\theta}\, e^{-\frac{x}{\theta}}\mathbf{1}_{(0,\infty)}(x), \quad x \in \mathbb{R},$$

for some $\theta > 0$, and let $\overline{X}$ denote the sample mean.

- **a.** Compute $E_\theta(\overline{X}^2)$.
- **b.** Find the uniform minimum variance unbiased estimator of $\varphi(\theta) = \mathrm{Var}_\theta(X_1)$.
- **c.** Check for $n = 1$ whether the estimator in part **b** achieves the Cramér-Rao lower bound.

3. Let $X_1, \dots, X_8$ be independent and identically distributed random variables, and suppose that each X_i has a standard normal distribution. Define $\overline{X}_1 = \frac{1}{4}\sum_{i=1}^{4} X_i$ and $\overline{X}_2 = \frac{1}{4}\sum_{i=5}^{8} X_i$.

- **a.** What is the distribution of $\frac{1}{2}(\overline{X}_1 + \overline{X}_2)$?
- **b.** What is the distribution of $4\overline{X}_1^2$?
- **c.** What is the distribution of $\overline{X}_1^2/\overline{X}_2^2$?
- **d.** For a certain number $c > 0$ the random variable $\overline{X}_1/\sqrt{c\overline{X}_2^2}$ has a student-type distribution. What is c and what is the number of degrees of freedom?

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4. Given is a random sample X_1, X_2, X_3 of size 3 from a Normal $(\theta, 1)$ distribution, $\theta \in \mathbb{R}$.

a. Find a sufficient and complete statistic for $\theta \in \mathbb{R}$.

b. Find the uniform minimum variance unbiased estimator of $\varphi(\theta) = \theta^2$. Explain!

c. Determine $E(X_1^2 - \frac{1}{2}(X_2 - X_3)^2 | \overline{X})$, where $\overline{X}$ is the sample mean. Explain!

5. Suppose that $X_1, \ldots, X_n$ are independent random variables and that X_i has density

$$f_\theta(x) = \frac{1}{\theta t_i}\, e^{-\frac{x}{\theta t_i}} \mathbf{1}_{(0,\infty)}(x), \quad x \in \mathbb{R},$$

where $\theta > 0$ is an unknown parameter and $t_i \neq 0$ a given number $(i = 1, \ldots, n)$.

a. Determine the maximum likelihood estimator $\hat{\theta}$ of θ.

b. Determine the family of likelihood ratio tests for testing the null hypothesis $H_0 : \theta = 1$ against the alternative $H_1 : \theta \neq 1$, and prove that these tests are equivalent with rejecting H_0 when either $\hat{\theta} \leq c_1$ or $\hat{\theta} \geq c_2$ for suitable numbers $c_1 \leq c_2$ ($\hat{\theta}$ as in **a**).

6. Consider the function

$$f(x, y) = cx^2 y \mathbf{1}_D(x, y), \quad (x, y) \in \mathbb{R}^2,$$

where $D = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 < y < 1\}$.

a. Determine the number c, such that f is a p.d.f.

Henceforth let c be the number obtained in **a** and suppose that X, Y are random variables with joint p.d.f. f.

b. Compute the marignal p.d.f. f_X of X.

c. Compute the conditional expectation $E(Y|X)$ of Y given X.

d. Which of $E(Y|X)$ and Y has the smaller variance?

7. Let $X_1, \ldots, X_{100}$ be independent and identically distributed random variables, and let each X_i have p.d.f.

$$f(x) = 2x\mathbf{1}_{(0,1)}(x), \quad x \in \mathbb{R}.$$

Give an approximation for the probability that at least 20 of these random variables exceed $2/\sqrt{5}$.

8. Let X and Y be independent and identically distributed, each with p.d.f.

$$f_\theta(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \mathbf{1}_{(0,\infty)}(x), \quad x \in \mathbb{R},$$

where $\theta > 0$ is an unknown parameter.

a. Argue that $X/(X+Y)$ is ancillary for $\theta > 0$ (i.e. has a distribution that does not depend on θ).

b. Argue that $X+Y$ and $X/(X+Y)$ are stochastically independent for each $\theta > 0$.