Statistics Prelim, August 2007

Work all 7 problems. Please begin each problem on a new sheet of paper, and do not use both sides of the paper (continue on a new sheet of paper if the solution will not fit on a single sheet). You may use a calculator.

1. Let $X_1, \ldots, X_{10}$ be a random sample of size 10 from an exponential distribution with mean μ. Recall that $f(x \mid \mu) = \frac{1}{\mu} e^{-x/\mu}; x > 0, \mu > 0$.
   (a) Find a complete sufficient statistic for μ.
   (b) Derive an exact 95% confidence interval for μ.

2. Two players, A and B, play a game. Two fair six-sided dice are rolled repeatedly. The first time a total of 10 appears, Player A wins, while the first time a total of 6 appears, Player B wins. The game stops as soon as one of the players wins.
   (a) Compute the probability that player A wins.
   (b) What is the expected number of rolls for a game?

3. Let $X_1, \ldots, X_n$ be a random sample of size n from the density function $f(x \mid \theta) = \frac{x+1}{\theta(\theta+1)} e^{-x/\theta}; x > 0, \theta > 0$.
   (a) Find $E[X_i]$.
   (b) Find a complete sufficient statistic for θ.
   (c) Find the maximum likelihood estimator for θ.

4. Let $X_1, \ldots, X_n$ be a random sample of size n from the density function $f(x \mid \theta) = e^{-(x-\theta)}; x \geq \theta, \theta \in \Re$. Consider the hypothesis test $H_0: \theta \leq \theta_0$ vs. $H_1: \theta > \theta_0$.
   (a) Give the form of the likelihood ratio test.
   (b) If $n = 38$ and $x_{(1)} = 5.004$, test $H_0: \theta \leq 4.5$ vs. $H_1: \theta > 4.5$ at a level $\alpha = 0.05$. The test may be either exactly level 0.05 or approximately level 0.05.

5. Let $X_1, \ldots, X_n$ be a random sample of size n from a $U(\theta-½, \theta+½)$ distribution (continuous uniform) with $\theta \in \Re$.
   (a) Find the maximum likelihood estimator for θ.
   (b) Suppose that, for $\lambda > 0$, the prior distribution for θ is $U(-\lambda, \lambda)$. Find the posterior distribution of θ given $x_1, \ldots, x_n$.
   (c) Find the expected value of the posterior distribution, which is the Bayesian estimator for θ under squared-error loss.
   (d) What issues can you see with using the prior given in part (b)?

6. Let X_1, X_2, X_3 be independent and identically distributed with pdf $f(x) = e^{-x}; x > 0$.
Let $Y_1 = X_1$, $Y_2 = X_1 + X_2$, and $Y_3 = X_1 + X_2 + X_3$.
   (a) Find the joint pdf of Y_1, Y_2, and Y_3.
   (b) Find the marginal distribution of Y_3.
   (c) Find the joint distribution of $Y_1, Y_2 \mid Y_3 = y_3$.

   Recall that if $\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then the determinant of $\mathbf{A}$ is given by
   $\det(\mathbf{A}) = aei + bfg + cdh - gec - hfa - idb$.

7. Let $X_1, \ldots, X_n$ be a random sample of size n from the density function
   $f(x \mid \alpha) = \dfrac{\alpha^{\alpha} x^{\alpha-1} e^{-\alpha x}}{\Gamma(\alpha)}; x > 0, \alpha > 0$.
   (a) Write down the likelihood function.
   (b) Show that there is a sufficient statistic such that the likelihood has the monotone likelihood ratio property with respect to that sufficient statistic.
   (c) Determine the uniformly most powerful test for testing H_0: $\alpha \leq 1$ vs. H_1: $\alpha > 1$.