Statistics Prelim, May 2007
Work all 8 problems. Please begin each problem on a new sheet of paper, and do not use both sides of the paper (continue on a new sheet of paper if the solution will not fit on a single sheet).

1. Let X_1 be an observation from a $N(\mu, \sigma^2)$ population, and let X_2 be an observation from a $N(\mu, k\sigma^2)$ population, where k is a known positive constant and X_1 and X_2 are independent.
   (a) Find a complete and sufficient statistic for (μ, σ^2).
   (b) Find the minimum variance unbiased estimator of μ.
   (c) Show that $\bar{X} = (X_1 + X_2)/2$ is an unbiased estimator of μ. Compute the variance of $\bar{X}$ and the estimator found in part (b). For what value of k will the two variances be equal?

2. Let $X_1, \ldots, X_n$ be a random sample ***with replacement*** from the integers $1, 2, \ldots, N$. Obtain the most powerful test for H_0: $N = N_0$ versus H_1: $N = N_1$, where $N_1 > N_0$.

3. Let X and Y be independent continuous random variables. Prove that

   $P(X < Y) = \int_{-\infty}^{\infty} F_x(y) f_y(y)dy$, where F_X is the cdf of X and f_Y is the pdf of Y.

4. Suppose X_1, X_2, and X_3 are independent and identically distributed exponential random variables with mean $\beta > 0$. Find $P(X_3 > \min(X_1, X_2))$.

5. On observing X from a normal distribution with mean μ and variance 1, a statistician declares the confidence interval $[X-1.96, X+1.96]$. A Bayesian has the prior belief that μ is normal with mean 0 and variance 1/3. Compute the posterior probability, as a function of X, that μ is in the interval.

6. Let the random vector (X, Y) have the joint density

   $f(x, y \mid \alpha, \beta, \gamma) = k(\alpha, \beta, \gamma) x^{\alpha-1} y^{\beta-1} (1 - x - y)^{\gamma-1}; x > 0, y > 0, x + y < 1, \alpha, \beta, \gamma > 0,$

   where $k(\alpha, \beta, \gamma)$ is a normalizing constant such that the density integrates to 1.
   (a) Find the marginal distribution of Y. (Hint: Make the transformation $u = \dfrac{x}{1-y}$.)
   (b) Find the conditional density of X given Y.

7. Suppose $X_1, \ldots, X_n$ is a random sample from

$$f(x \mid \beta, \mu) = \frac{1}{\beta} \exp\left[\frac{-(x-\mu)}{\beta}\right]; x > \mu, \mu > 0, \beta > 0.$$

(a) Find the method-of-moments estimators of μ and β.
(b) Find the maximum likelihood estimators of μ and β.

8. Suppose $X_1, \ldots, X_n$ is a random sample from a $N(\mu, \sigma^2)$ population. Consider the test of H_0: $\mu = 0$ vs. H_1: $\mu \neq 0$. Show that the likelihood ratio test is equivalent to the usual t-test, which is to reject H_0 if $\left|\frac{\sqrt{n}\bar{X}}{S}\right| > t_{\alpha/2;n-1}$, where S is the sample standard deviation, and $t_{\alpha/2;n-1}$ is such that $P(T_{n-1} > t_{\alpha/2;n-1}) = \alpha/2$ with T_{n-1} a t-random variable with n-1 degrees of freedom.