Statistics Prelim, May 2008

Work all 6 problems. Please begin each problem on a new sheet of paper, and do not use both sides of the paper (continue on a new sheet of paper if the solution will not fit on a single sheet). Calculators are not allowed.

1. Suppose that X_1 and X_2 are independent and identically distributed (i.i.d.) random variables with the density function

   $f(x) = \frac{1}{x^2} I_{(1,\infty)}(x).$

   Let $Y_1 = X_1/(X_1 + X_2)$ and $Y_2 = X_1 + X_2$.

   (a) Find the joint density function of Y_1 and Y_2.

   (b) Find the marginal density functions of Y_1 and Y_2.

   (c) Find $\mathrm{E}\left[Y_2 | Y_1 = \frac{1}{4}\right]$.

2. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the population described by the density function

   $f(x|\theta) = \frac{2x}{\theta} \exp\left(-\frac{x^2}{\theta}\right); x > 0, \theta > 0.$

   Recall that the gamma function is defined as $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

   (a) Find the method of moments (MoM) estimator for θ.

   (b) Find the maximum likelihood estimator (MLE) for θ.

3. Let X_1, X_2 be a random sample of size 2 from the population described by the density function

   $f(x|\theta) = \frac{2x}{\theta} \exp\left(-\frac{x^2}{\theta}\right); x > 0, \theta > 0.$

   (Please note that unlike the previous problem, the sample size is 2 here!)

   (a) Find the uniform minimum variance unbiased estimator (UMVUE) for θ.

   (b) Let $W_1 = X_1/X_2$ and $W_2 = X_1^2 + X_2^2$. Use sufficiency and completeness arguments to show that W_1 and W_2 are independent.

4. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the population described by the density function

   $f(x|\theta) = e^{-(x-\theta)} I_{(\theta,\infty)}(x).$

   (a) Construct a $(1-\alpha) \times 100\%$ confidence interval for θ based on the sufficient statistic for θ.

   (b) Construct a $(1-\alpha) \times 100\%$ confidence interval for θ based on $\bar{X}$.

   (c) Which do you prefer? Why?

   (Note: If the quantiles of any particular distribution are required, simply state how they would be found.)

5. Let $X_1, X_2, \dots, X_n$ be a random sample of size n from the population described by the density function

$$f(x|\theta) = \frac{1}{\theta} x^{(1-\theta)/\theta}; 0 < x < 1, \theta > 0.$$

(a) Show that the sufficient statistic has monotone likelihood ratio (MLR).

(b) If we are interested in testing $\text{H}_0: \theta \leq \theta_0$ vs. $\text{H}_1: \theta > \theta_0$, derive the form of the uniformly most powerful (UMP) level-α test. Give reasons for any claims you make.

(c) If $X_1, X_2, \dots, X_n$ have the distribution given above, then $-\ln(X_i)$ has an exponential distribution with mean θ. Derive an expression for the value of the power function $\beta(\theta)$ that can be evaluated by using standard statistical tables (i.e. chi-squared, F, standard normal, or t tables).

(d) If someone approached you with another test for the same hypotheses and this new test has size less-than or equal-to α, what can you say about the value of the power function of this new test at $\theta = \theta_1 > \theta_0$?

6. Suppose that a <u>fair</u> coin is flipped n times, where n is unknown. All that is known is the number of times the coin came up "heads", X. It is of interest to test $\text{H}_0: n = n_0$ vs. $\text{H}_1: n = n_1$, where $n_0 < n_1$.

(a) State the distribution of X. Specify the values of any known parameters.

(b) Find the uniformly most powerful (UMP) level-α test for the above hypotheses. Give reasons for any claims you make.

(c) Is it possible to find a size-α test for an arbitrarily chosen α? Why or why not?

(d) Is the rejection region found in part (b) also the rejection region for the UMP level-α test of $\text{H}_0: n \leq n_0$ vs. $\text{H}_1: n > n_0$? Why or why not?