Statistics Prelim, August 2009

Work all 6 problems. Begin each problem on a new sheet of paper, and do not use both sides of the paper (continue on a new sheet of paper if the solution will not fit on a single sheet). Calculators are not allowed. State any theorem or fact you use. You may need the following probability distributions for problems.

- $Poisson(\lambda): \ f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \cdots$
- $Exp(\lambda): \ f(x) = \lambda e^{-\lambda x}, \quad x > 0$
- $Beta(\alpha, \beta)$
$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1$$
- $Gamma(\alpha, \beta)$
$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \quad x > 0$$

1. Let the random variables X and Y have joint density function
$$f_{X,Y}(x,y) = \begin{cases} ye^{-y(x+1)}, & for \ \ 0 < x, \ 0 < y \\ 0, & elsewhere \end{cases}$$
   (a) Determine the marginal density f_X of X.
   (b) Are X and Y independent? Justify your answer.
   (c) For $p \in \mathbb{R}$, compute $E\left[(X+1)^{-p}\right]$.
   (d) Find the conditional density of Y given $X = x$.
   (e) Find the conditional mean and variance of Y given $X = x$.

2. Suppose that $X|\lambda \sim Poisson(\lambda)$, and $\lambda \sim Gamma(\alpha, \beta)$.
   (a) Find the marginal probability mass function (pmf) of X.
   (b) Find the marginal mean and variance of X.
   (c) Suppose that $X_1, \cdots, X_n$ are an IID sample from the marginal pmf of X in (a). Find the Method of Moment estimator of (α, β).
   (d) Suppose X_1, X_2 are an IID sample from the following pmf
$$P_\beta(X = x) = \frac{\Gamma(2+x)}{\Gamma(2)x!\beta^2}\left(\frac{\beta}{\beta+1}\right)^{x+2}, \qquad \text{for } x = 0, 1, 2, \cdots.$$
   Find the UMP test for testing $H_0: \beta = 1$ against $H_1: \beta > 1$ with significance level $\alpha = 13/16$.

3. Let $X_1, X_2, \cdots, X_n$ be a random sample from a population with probability density function:

$$f(x;\theta) = \begin{cases} 2\theta x e^{-\theta x^2}, & x > 0 \\ 0, & elsewhere \end{cases}$$

where $\theta > 0$. You may use the fact that $E(X^2) = 1/\theta$, and $Var(X^2) = 1/\theta^2$.

(a) Show that $\hat{\theta} = n/\sum_{i=1}^{n} X_i^2$ is a consistent estimator of θ.

(b) Find the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$.

(c) Find the asymptotic relative efficiency of $\hat{\theta}$ using the Cramer-Rao lower bound.

4. Suppose that $X_1, \cdots, X_n$ is an IID random sample from a $Poisson(\lambda)$.

(a) Find the MLE of λ.

(b) Find the MLE of $\eta = P_\lambda(X_1 = 2)$.

(c) Find the MVUE of η in (b).

5. Suppose X is a random variable with probability mass function $p(x;\theta)$, where $\theta = 0$ or 1. The probability distribution for X is given by

x	0	1	2	3	4	5
$p(x;\theta=0)$	0.05	0.05	0.1	0.1	0.2	0.5
$p(x;\theta=1)$	0.1	0.1	0.25	0.2	0.25	0.1

(a) Find the most powerful test of size $\alpha = 0.1$ for testing $\theta = 0$ against $\theta = 1$, when we observe the single random value X_1.

(b) Compute the power for the test found in (a).

(c) Suppose that $X_1 = 3$, $X_2 = 5$ are the observed random sample. What is the MLE of θ?

6. Suppose that we have two independent random samples: $X_1, \cdots, X_n \overset{IID}{\sim} Beta(\mu, 1)$, and $Y_1, \cdots, Y_m \overset{IID}{\sim} Beta(\theta, 1)$.

(a) Find the likelihood ratio test for $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

(b) Show that the test in (a) can be based on the statistic

$$T = \frac{\sum_{i=1}^{n} \log X_i}{\sum_{i=1}^{n} \log X_i + \sum_{j=1}^{m} \log Y_j}$$

(c) Find the distribution of T when H_0 is true, and then explain how to construct a test of size $\alpha = 0.1$.