Statistics Prelim, May 2009

Work all 6 problems. Please begin each problem on a new sheet of paper, and do not use both sides of the paper (continue on a new sheet of paper if the solution will not fit on a single sheet). Calculators are not allowed. You may need the following probability distributions for problems.

- $b(n,p)$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x = 0, 1, \cdots, n$$

- $Poisson(\lambda)$

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0, 1, \cdots$$

- $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < \infty$$

- $Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0$$

1. Let Y be a random variable distributed as $Poisson(\lambda)$ and suppose that the conditional distribution of the random variable X, given $Y = y$, is $b(y, p)$.

   (a) Find the marginal probability mass function of X.

   (b) Find the conditional probability mass function of Y given X.

2. Suppose X_1, X_2 and X_3 are IID random variables from $Exp(\lambda)$. Let $Z_1 = 3Y_1$, $Z_2 = 2(Y_2 - Y_1)$ and $Z_3 = Y_3 - Y_2$, where $Y_1 < Y_2 < Y_3$ are the order statistics corresponding to X_1, X_2, X_3.

   (a) Find the joint pdf of (Z_1, Z_2, Z_3).

   (b) Show that Z_1, Z_2 and Z_3 are independent and find their marginal distributions.

3. Suppose $Y_i \sim N(\beta x_i, 1)$ where the x_i's are known constants and β is an unknown parameter. Assume the Y_i's are independent.

   (a) Find the MLE $\hat{\beta}$ of β.

   (b) Show that $\hat{\beta}$ is an unbiased estimator of β

   (c) Find the MSE of $\hat{\beta}$.

4. Let $X_1, \cdots, X_n$ be a sample from a population with the density

$$f(x;\theta) = \theta(\theta+1)x^{\theta-1}(1-x),$$

where $0 < x < 1$, $\theta > 0$.

(a) Show that

$$T_n = \frac{2\bar{X}}{1-\bar{X}}$$

is a method of moments estimate of θ.

(b) Show that

$$\sqrt{n}(T_n - \theta) \xrightarrow{d} N(0, \theta(\theta+2)^2/2(\theta+3))$$

(c) Show that T_n is not asymptotically efficient.

5. Let $X_1, \cdots, X_n$ be a random sample from $N(\mu_1, \sigma^2)$ and let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu_2, 4\sigma^2)$.

(a) Find the maximum likelihood estimators of μ_1, μ_2, and σ^2.

(b) Derive the likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$.

(c) What is the sampling distribution of your test statistic in (b) under H_0?

6. Suppose that $X_1, \cdots, X_n$ is a random sample from $Exp(\theta)$ and that $Y_1, \cdots, Y_m$ is a random sample from $Exp(\mu)$. Assume X_i and Y_j are independent for any i and j.

(a) Construct the likelihood ratio test that depends on the statistic

$$T = \frac{\sum_{i=1}^{n} X_i}{\sum_{j=1}^{m} Y_j}$$

for testing $H_0 : \theta = \mu$ against $H_1 : \theta \neq \mu$.

(b) Find the distribution of T under H_0.

(c) Suppose that $n = m = 1$. Give the exact rejection region for the size 0.05 LRT for testing $H_0 : \theta = \mu$ against $H_1 : \theta \neq \mu$.