# Statistics Prelim, August 2010

Work all 7 problems. Begin each problem on a new sheet of paper, and do not use both sides of the paper (continue on a new sheet of paper if the solution will not fit on a single sheet). Calculators are not allowed. State any theorem or fact you use. You may need the following probability distributions for problems.

$$Poisson(\lambda) : f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots, \quad \lambda > 0$$

$$Exp(\lambda) : f(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0$$

$$b(n,p) : f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n, \quad 0 < p < 1$$

$$N(\mu, \sigma^2) : f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma^2 > 0$$

$$\chi^2(d) : f(x) = \frac{1}{\Gamma(d/2) 2^{d/2}} x^{d/2-1} e^{-x/2}, \quad x > 0, \quad d > 0$$

$$where \ \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

1. (15 points) Let $Z_1$ and $Z_2$ be independent $Exp(\lambda)$ random variables, $\lambda > 0$. Define $X = Z_2$ and $Y = Z_1 + Z_1 Z_2$.
   (a) Find the joint density of $X$ and $Y$.
   (b) Find $E(Y|X = x)$.
   (c) Find $\text{Var}(E(Y|X))$.

2. (15 points) Suppose $X_1, \ldots, X_n$ is a random sample from the probability density function
$$f(x;\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0.$$
   Let $W_i = -\log(X_i)$ and let $\theta$ be the unknown parameter.
   (a) Show that $\sum_{i=1}^n W_i$ is a complete and sufficient statistic for $\theta$.
   (b) Show that the distribution of $2\theta \sum_{i=1}^n W_i$ is $\chi^2(2n)$.
   (c) Find the MVUE of $\theta$. (Hint: Calculate $E\left[(\sum_{i=1}^n W_i)^{-1}\right]$.)

3. (15 points) Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent copies of $(X, Y)$, whose joint distribution is specified as follows: the marginal distribution of $X$ is $Poisson(\lambda)$, and conditioning on $X = x$, $Y$ is distributed as $b(x+1, p)$.
   (a) Show that the covariance between $X$ and $Y$ is $\alpha = p\lambda$.
   (b) Find the maximum likelihood estimator of $\alpha$, call it $\hat{\alpha}$.
   (c) Find the asymptotic distribution of $\sqrt{n}(\hat{\alpha} - \alpha)$.

4. (10 points) If $X_n \sim b(n, 1/n)$, show that the limiting distribution of $X_n$ is $Poisson(1)$.