Statistics Prelim, May 2010

Work all 7 problems. Begin each problem on a new sheet of paper, and do not use both sides of the paper (continue on a new sheet of paper if the solution will not fit on a single sheet). Calculators are not allowed. State any theorem or fact you use. You may need the following probability distributions for problems.

$$Poisson(\lambda): f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots, \quad \lambda > 0$$

$$Exp(\lambda): f(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0$$

$$N(\mu, \sigma^2): f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma^2 > 0$$

$$U[a, b]: f(x) = \frac{1}{b-a}, \quad a \le x \le b, \quad a < b$$

1. (15 points) Suppose X_1 and X_2 are independent $Exp(\lambda)$ random variables. Let $Y_1 = X_1 - X_2$ and $Y_2 = X_2$.
   (a) Find the joint density of Y_1 and Y_2.
   (b) Find the marginal density of Y_1.

2. (15 points) Let $X_0, X_1, \ldots, X_n$ be independent and identically distributed $N(0, 1)$ random variables and $Y_1, \ldots, Y_n$ be independent and identically distributed $U[0, 1]$ random variables. Assume X_i and Y_j are independent for any i and j. Define $W_i = Y_i X_i + (1 - Y_i)X_0$, $i = 1, \ldots, n$.
   (a) Find the mean and variance of W_i.
   (b) Find the covariance of W_i and W_j for $i \neq j$.
   (c) Find the mean and variance of $S = \sum_{i=1}^n W_i$.

3. (15 points) Suppose $X_1, \ldots, X_n$ are independent discrete random variables with probability mass function for X_i, $i = 1, \ldots, n$, given by

x	1	2	3
$p(x;\theta)$	θ^2	$2\theta(1-\theta)$	$(1-\theta)^2$

   with $0 < \theta < 1$. Let N_j be the number of observations equal to j, $j = 1, 2, 3$.
   (a) Show that $U = 2N_1 + N_2$ is a sufficient statistic for θ.
   (b) Find the distribution of U.
   (c) Find the MVUE of θ.

4. (10 points) Suppose X_1, X_2 are independent discrete random variables with probability mass function for X_i, $i = 1, 2$, given by

x	0	1	2
$p(x;\theta)$	$e^{-\theta}$	$\theta e^{-\theta}$	$1 - e^{-\theta} - \theta e^{-\theta}$

   with $0 < \theta$. Show that $X_1 + X_2$ is *not* a sufficient statistic for θ.

5. (15 points) It is sometimes reasonable to think of the lifetime of a piece of equipment as a random variable following an $Exp(1/\lambda)$ distribution where λ is an unknown positive real number. Suppose n identical pieces of equipment are to be run until they fail and the (random) failure times are given by $X_1, \ldots, X_n$. Assume that these are a random sample from the $Exp(1/\lambda)$ distribution. Consider the probability of early failure given by

$$\eta = P_\lambda(X_1 < x) = 1 - \exp(-x/\lambda)$$

for some fixed $x > 0$.

(a) Find the MLE of η.

(b) Find the MVUE of η. (Your answer must be a closed form function of the data.)

6. (15 points) Let $X_1, \ldots, X_n$ be an IID sample from $N(\mu, \sigma^2)$, where $\mu \neq 0$ and $\sigma^2 > 0$. Let $\bar{X}_n$ be the sample average and let S^2 be the sample variance.

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

(a) Show that $\bar{X}_n \xrightarrow{P} \mu$ and $S_n^2 \xrightarrow{P} \sigma^2$.

(b) Find the joint asymptotic distribution of

$$\sqrt{n}\left(\begin{pmatrix}\bar{X}_n \\ S_n^2\end{pmatrix} - \begin{pmatrix}\mu \\ \sigma^2\end{pmatrix}\right).$$

(Hint: $E(X_i^3) = \mu^3 + 3\mu\sigma^2$ and $E(X_i^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$)

(c) Find the asymptotic distribution of $\sqrt{n}\,(S_n^2/\bar{X}_n - \sigma^2/\mu)$.

7. (15 points) Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the following density function

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} x^{(1-\theta)/\theta}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where $\theta > 0$.

(a) Show that the distribution of $-\log X_i$ is $Exp(1/\theta)$.

(b) Show that this family of distributions has monotone likelihood ratio in the statistic $T = -\sum_{i=1}^{n} \log X_i$.

(c) If we are interested in testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$, derive the form of the UMP size α test.

(d) Explain how you can construct the UMP size $\alpha = 0.05$ test by using standard statistical tables such as Normal, t, F, and χ^2 tables.