Probability and Statistics Preliminary Examination: May 2016

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I_{[x\in A]}$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $E(X)$: expectation of random variable X.
  - $V(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a,b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

Poisson(λ): $\mathbb{E}(X)=\lambda$, $\mathbb{V}(X)=\lambda$, and pmf and mgf given respectively by

$$f(x)=\frac{e^{-\lambda}\lambda^x}{x!}I(x\in\{0,1,\cdots\}), \qquad M(t)=\exp\{\lambda(e^t-1)\}$$

Geometric(p): $\mathbb{E}(X)=1/p$, $\mathbb{V}(X)=(1-p)/p^2$, and pmf and mgf given respectively by

$$f(x)=p(1-p)^{x-1}I(x\in\{1,2,\cdots\}), \qquad M(t)=\frac{pe^t}{1-(1-p)e^t}, \quad t<-\log(1-p)$$

Negative-Binomial(r,p): $\mathbb{E}(X)=r(1-p)/p$, $\mathbb{V}(X)=r(1-p)/p^2$, and pmf and mgf, respectively:

$$f(x)=\binom{r+x-1}{x}p^r(1-p)^xI(x\in\{0,1,\cdots\}), \qquad M(t)=\left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t<-\log(1-p)$$

Gamma(α,β): $\mathbb{E}(X)=\alpha\beta$, $\mathbb{V}(X)=\alpha\beta^2$, and pdf and mgf given respectively by

$$f(x)=\frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}I(x>0), \qquad M(t)=\left(\frac{1}{1-\beta t}\right)^\alpha, \quad t<1/\beta$$

Order Statistics: Let $X_{(1)}\le\cdots\le X_{(n)}$ denote the order statistics from a random sample $X_1,\ldots,X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)},X_{(j)})$, and $(X_{(1)},\ldots,X_{(n)})$, are given by:

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j}f(x)I(-\infty<x<\infty)$$

$$f_{X_{(i)},X_{(j)}}(x_i,x_j)=\frac{n!}{(i-1)!(j-1-i)!(n-j)!}f(x_i)f(x_j)[F(x_i)]^{i-1}[F(x_j)-F(x_i)]^{j-1-i}[1-F(x_j)]^{n-j} \times I(-\infty<x_i\le x_j<\infty)$$

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n)=n!f(x_1)\cdots f(x_n)I(-\infty<x_1\le\cdots\le x_n<\infty)$$

1. A deck of 40 cards has denominations $1, 2, \ldots, 10$, where there are 4 of each denomination. The cards are unsuited, so, for example, all of the 2s are identical. Suppose 5 cards are drawn from the deck behind a screen and only the results are told to you. There are two dealers that will be assigned to your game randomly. Dealer A will draw cards randomly without replacement. Dealer B will randomly draw cards with replacement. The dealer for your game is unknown to you. Let X denote the number of 2s drawn among the five draws.

   (a) Given that the first card is a 2, calculate the probability that the second card is also a 2.

   (b) Derive the pmf of X.

   (c) Suppose the same dealer is used for 10 independent games. If two or fewer 5s are drawn on exactly 8 of the trials, derive an expression for the probability that Dealer A was chosen for your games.

2. Let $X_1, \ldots, X_n$ be independent random variables, where $X_i \sim$Poisson(i), for $i = 1, \ldots, n$.

   (a) Derive the pdf for $\overline{X}$

   (b) Show that the pmf for $X_{(n)}$ is as follows:

$$P(X_{(n)} = x) = \exp\left(\frac{n(n+1)}{2}\right)\frac{(n!)^x}{(x!)^n}$$

   (c) Show that the nth moment of $X_{(n)}$ is $n!$.

3. Let Y be a random variable with pdf $f_Y(y) = (8y - 4)I_{[0.5<y<1]}$ and let X be a random variable such that $X|Y = y$ has pdf $f(x|y) = yx^{y-1}I_{[0<x<1]}$

   (a) Calculate $E[f(X|Y)]$.

   (b) Calculate $\rho_{X,Y}$, the correlation between X and Y.

4. Let $X_1, \ldots, X_n$ be independent random variables, where $X_i \sim$Gamma(α_i, β), for $i = 1, \ldots, n$.

   (a) Find a MSS for $(\alpha_1, \ldots, \alpha_n, \beta)$.

   (b) If $\alpha_1, \ldots, \alpha_n$ are known, derive an **exact** $100(1-\alpha)\%$ confidence interval for β.

   (c) If $\alpha_1, \ldots, \alpha_n$ are known, find the UMVUE for β.

   (d) If $\alpha_1 = \cdots = \alpha_n = \alpha$ is known, derive the asymptotic distribution of the MLE of the mode of Gamma(α, β).

5. Let $X_1, \ldots, X_n$ $(n > 2)$ be a iid Uniform$(\theta, \theta + 1)$.

   (a) Derive the MOME for θ and its MSE.

   (b) Construct an asymptotic level $1 - \alpha$ test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

   (c) Derive the Bayes estimator for θ and its probability distribution if $\pi(\theta)$ is Uniform(a, b).