Probability and Statistics Preliminary Examination: August 2016

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I_{[x\in A]}$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $E(X)$: expectation of random variable X.
  - $Var(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  Poisson(λ): $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  Geometric(p): $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

$$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$$

  Negative-Binomial(r, p): $\mathbb{E}(X) = r(1-p)/p$, $\mathbb{V}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x I(x \in \{0, 1, \cdots\}), \qquad M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

  Gamma(α, β): $\mathbb{E}(X) = \alpha\beta$, $\mathbb{V}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < 1/\beta$$

  Order Statistics: Let $X_{(1)} \le \cdots \le X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} [F(x)]^{j-1} [1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)},X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-1-i} [1-F(x_j)]^{n-j} \times I(-\infty < x_i \le x_j < \infty)$$

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \le \cdots \le x_n < \infty)$$

1. An urn contains M balls. R are red and G are green. Suppose you draw balls without replacement successively until you have drawn at least one ball of each color, at which point you stop. Let K be the random variable denoting the number of draws you made.

   (a) Assuming $M > 3$, such that $R > 2, G > 2$, and $R + G = M$, calculate $P(K = 2)$ and $P(K = 3)$ up to a multiplicative constant c that results in

   $$\sum_{i=1}^{\min(R,G)+1} P(K = i) = 1.$$

   (b) If this process is independently repeated $n > 2$ times, derive an expression for the probability that at least 2 of the stopping times are greater than 3.

2. Let $X_1 \sim$Uniform$(0, 1)$, $X_2|X_1 \sim$Uniform$(0, X_1)$, and $X_3|X_2 \sim$Uniform$(0, X_2)$.

   (a) Derive the marginal pdf for X_3.

   (b) Calculate $\mathrm{Cov}(X_1, X_2)$, $\mathrm{Cov}(X_1, X_3)$, and $\mathrm{Cov}(X_2, X_3)$

3. Let $X_1 \sim$Poisson(λ), and $X_i|X_{i-1} = x_{i-1} \sim$Poisson$(\lambda x_{i-1})$ for $i = 2, \ldots, n$.

   (a) Find the MSS for λ.

   (b) Derive the MLE for λ.

   (c) Find the Cramér-Rao lower bound for an unbiased estimator for λ if $n = 3$. Use this to derive an expression for the Cramér-Rao lower bound in the same scenario for a general value of n.

4. Let $X_1, \ldots, X_n$ be iid random variables from the following pdf:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-1}{2\sigma^2}(\log(x) - \mu)^2\right) I_{[0<x]} I_{[0<\sigma]}.$$

Let $\tilde{X} = \left(\prod_{i=1}^n X_i\right)^{1/n}$ denote the geometric sample mean. Also, let $Y_i = \log(X_i)$ for $i = 1, \ldots, n$.

(a) Find the distribution of

$$\frac{9(Y_1 - Y_2)^2}{2(Y_n - 2Y_1 - 2Y_2)^2}$$

(b) Derive the distribution of $\tilde{X}$.

(c) Find the asymptotic distribution of

$$n\left(\exp\left(-(\log(\tilde{X}) - \mu)^2\right) - 1\right)$$

5. Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ random variables, where $\mu > 0$.

(a) Derive the test statistic for a level α LRT of $H_0 : \sigma^2 = \mu$ vs. $H_1 : \sigma^2 \neq \mu$.

(b) If $\mu \sim N(\gamma, \tau^2)$, derive a $100(1-\alpha)\%$ credible interval for μ.

(c) In the same setting as part (*b*), find the PBE for μ^2 and its MSE if $\sigma^2 = \sigma_0^2$.