Probability and Statistics Preliminary Examination: August 2017

Instructions:

- Work all 6 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 6 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I_{[x\in A]}$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $E(X)$: expectation of random variable X.
  - $Var(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  Poisson(λ)**:** $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  Geometric(p)**:** $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

$$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$$

  Negative-Binomial(r, p)**:** $\mathbb{E}(X) = r(1-p)/p$, $\mathbb{V}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x I(x \in \{0, 1, \cdots\}), \qquad M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

  Gamma(α, β)**:** $\mathbb{E}(X) = \alpha/\beta$, $\mathbb{V}(X) = \alpha/\beta^2$, and pdf and mgf given respectively by

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1-t/\beta}\right)^\alpha, \quad t < \beta$$

  Order Statistics: Let $X_{(1)} \le \cdots \le X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} [F(x)]^{j-1} [1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)},X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1} [F(x_j)-F(x_i)]^{j-1-i} [1-F(x_j)]^{n-j} \times I(-\infty < x_i \le x_j < \infty)$$

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \le \cdots \le x_n < \infty)$$

1. A box containing 100 corks has been filled up by taking 100 corks at random from a lot of 1,000 corks which originally contained 10 defective corks. A cork is chosen at random from the foregoing box (that has 100 corks) and it was found to be defective.

   (a) Given that information, find the probability distribution of the number of defective corks in the box.

   (b) Given that the first cork drawn from the box was defective, what is the probability that another cork drawn from the remaining 99 in the box will also be defective?

2. Consider two standardized random variables, Z_1 and Z_2, i.e, $E(Z_i) = 0,\ Var(Z_i) = 1,\ \ i = 1, 2$ and correlation coefficient ρ.

   (a) Show that $E(max(Z_1^2, Z_2^2)) \leq 1 + \sqrt{1 - \rho^2}$.

   (b) Hence show that for the unstandardized random variables X_1 and X_2 with $E(X_i) = \mu_i,\ \ i = 1, 2$ and $Var(X_i) = \sigma_i^2,\ \ i = 1, 2$ and correlation coefficient ρ, the following holds for any $t > 0$

$$P(|X_1 - \mu_1| \geq t\sigma_1 \text{ or } |X_2 - \mu_2| \geq t\sigma_2) \leq \frac{1 + \sqrt{1 - \rho^2}}{t^2}$$

3. Let $X_1, X_2, ...X_n \overset{iid}{\sim} Normal(\mu, 1)$ with $\mu \geq 0$.

   (a) Find the MLE, $\hat{\mu}_{MLE}$, of μ

   (b) Find the asymptotic distribution of $\hat{\mu}_{MLE}$.

4. $X_1, X_2, ...X_n \overset{iid}{\sim} Discrete\ Uniform(1, N)$, i.e.

$$P(X_i = k) = \begin{cases} \frac{1}{N}, & k = 1, 2, ..., N \\ 0, & \text{otherwise} \end{cases}$$

.

   (a) Find the sufficient statistic for N. Show that it is also complete.

   (b) Find an UMVUE for N.

5. Consider the following two-sample problem, $X_1, X_2, ...X_m \overset{iid}{\sim} Poisson(\lambda_1)$ and $Y_1, Y_2, ...Y_n \overset{iid}{\sim} Poisson(\lambda_2)$, X_i, Y_j are mutually independent $\forall i = 1, 2, ..., m;\ j = 1, 2, ..., n$. We wish to test $H_0 : \lambda_1 = \lambda_2$ Vs. $H_1 : \lambda_1 \neq \lambda_2$ using likelihood ratio test statistic. If $(m, n) \to \infty$ such that $m/(m+n) \to \lambda,\ \ \lambda \in (0, 1)$, show that the asymptotic distribution of the $-2\log(LRT)$ is given by $(\sqrt{1 - \lambda}W - \sqrt{\lambda}Z)^2$ where W, Z are independent $Normal(0, 1)$ variates.

6. Let $Y_1, Y_2, ..., Y_n$ be iid observations from an arbitrary distribution with mean μ, variance σ^2.

   (a) Find the asymptotic distribution of $W_n = \frac{1}{\sqrt{n}}\left(\frac{(n-1)s^2}{\sigma^2} - n\right)$. State the assumption(s) you need to make.

   (b) Use W_n to construct an approximate large sample $100(1 - \alpha)\%$ confidence interval for σ^2.