# Probability and Statistics Preliminary Examination: May 2018

**Instructions:**

- Work all 8 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Problems 3-8 are equally weighted, while problems 1-2 carry *half* the weight of problems 3-8. Parts within a problem are not necessarily equally weighted.
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $\mathbb{P}(\cdot)$: probability.
  - $I_{[x\in A]}$ or $I_A(x)$: indicator function for set $A$; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbb{E}(X)$: expectation of random variable $X$.
  - $\mathbb{V}(X)$: variance of random variable $X$.
  - $X \sim \mathrm{N}(a, b)$: $X$ has a normal distribution with mean $a$ and variance $b$.
- Common distributions and other results.

  **Poisson**$(\lambda)$**:** $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

  $$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  **Geometric**$(p)$**:** $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

  $$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$$

  **Negative-Binomial**$(r, p)$**:** $\mathbb{E}(X) = r(1-p)/p$, $\mathbb{V}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

  $$f(x) = \binom{r+x-1}{x} p^r (1-p)^x I(x \in \{0, 1, \cdots\}), \qquad M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

  **Gamma**$(\alpha, \beta)$**:** $\mathbb{E}(X) = \alpha\beta$, $\mathbb{V}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

  $$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < 1/\beta$$

  **Order Statistics:** Let $X_{(1)} \le \cdots \le X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If $X_1$ is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

  $$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} [F(x)]^{j-1} [1 - F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

  $$f_{X_{(i)}, X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-1-i} [1 - F(x_j)]^{n-j} \times I(-\infty < x_i \le x_j < \infty)$$

  $$f_{X_{(1)}, \ldots, X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \le \cdots \le x_n < \infty)$$

1. A closet contains 13 different pairs of gloves. The 26 gloves are randomly arranged into 13 pairs.
   (a) Find the probability that all left-hand gloves are paired with right-hand gloves (not necessarily matching).
   (b) Find the probability that all of the gloves are arranged so that each left-hand glove is paired with its matching right-hand glove.

2. Suppose that $X \sim$ Gamma$(\alpha, 1)$ where $\alpha$ is a positive integer. Further suppose that $\alpha$ has a Geometric$(p)$ prior disribution. Find the posterior distribution of $\alpha$ given $X = x$.

3. Suppose that $(X, Y)$ is a continuous bivariate random vector with joint pdf given by:

$$f(x,y) = \begin{cases} 2e^{-x-y}, & 0 < x < y < \infty; \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find $\mathbb{P}(X + Y < 1)$.
   (b) Find the marginal pdf of $X$, $f_X(x)$.
   (c) Find the conditional distribution of $Y|X = x$, where $x > 0$.
   (d) Find the distribution of $U = X + Y$. (Hint: Let $V = X$.)

4. Let $X_n, n = 1, 2, \ldots$ be a sequence of discrete random variables such that for each $n$, $\mathbb{P}\left(X_n = \frac{1}{n}\right) = \frac{n+1}{5n}$ and $\mathbb{P}\left(X_n = 1 - \frac{1}{n}\right) = \frac{4n-1}{5n}$.
   (a) Calculate the moment generating function of $X_n$.
   (b) Find the limiting distribution of $X_n$ as $n \to \infty$.

5. Suppose that $Y_i = \beta x_i + \varepsilon_i, i = 1, \ldots, n$, where
   - $x_i$ are known constants,
   - $\varepsilon_1, \ldots, \varepsilon_n$ is a $\text{N}(0, \sigma^2)$ random sample,
   - $\sigma^2$ is a known constant, and
   - $\beta$ is a $\text{N}(\theta, \delta^2)$ random variable.

   (a) Show that $\sum_{i=1}^n Y_i x_i$ is a sufficient statistic for $\beta$.
   (b) Let $Q = \frac{\sum_{i=1}^n Y_i x_i}{\sum_{j=1}^n x_j^2}$. Find the distribution of $Q|\beta$. (Note that $Q$ is the MLE of $\beta$, which is a fact you are not being asked to prove.)
   (c) Find the posterior distribution of $\beta$ given $Q = q$.

6. Let $X_1, \ldots, X_n$ be an iid random sample from a $\text{N}(\mu, \sigma^2)$ population. (Note that $\mathbb{E}\left[(X_i - \mu)^4\right] = 3\sigma^4$, and that $\mathbb{E}(Q^m) = 2^m \frac{\Gamma(m+k/2)}{\Gamma(k/2)}$ for $Q$ having a chi-squared distribution with $k$ degrees of freedom and $m$ a positive integer.)
   (a) Find the UMVUE for $\sigma^4$.
   (b) Calculate the Cramér-Rao Lower Bound for the estimator found in (a).
   (c) Does the estimator found in (a) attain the Cramér-Rao Lower Bound? Justify your answer.

7. Let $X_1, \ldots, X_n$ be a random sample from a continuous Uniform$(\alpha, \beta)$ population such that $\alpha < 0 < \beta$.

   (a) Find the MLEs of $\alpha$ and $\beta$.

   (b) Consider testing $H_0 : \beta = -\alpha$ versus $H_1 : \beta \neq -\alpha$. Find the LRT test statistic, and give the form of the rejection region for the LRT.

8. Let $X_1, \ldots, X_n$ be a random sample from the density given by:

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)}, & x > \theta; \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Prove that the MLE of $\theta$ is $\hat{\theta}_n = X_{(1)}$.

   (b) Show that $\hat{\theta}_n$ is a consistent estimator of $\theta$.

   (c) Construct the UMP level-$\alpha$ test of $H_0 : \theta = 4$ versus $H_1 : \theta > 4$ based on $X_1, \ldots, X_n$.