Probability and Statistics Preliminary Examination: August 2018

Instructions:

- Work all 6 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. All problems are equally weighted, but parts within a problem are not necessarily equally weighted.
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $\mathbb{P}(\cdot)$: probability.
  - $I_{[x\in A]}$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbb{E}(X)$: expectation of random variable X.
  - $\mathbb{V}(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  Poisson(λ)**:** $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

  $$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  Geometric(p)**:** $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

  $$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$$

  Beta(α, β)**:** $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathbb{V}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, and pdf and mgf, given respectively by:

  $$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0 \le x \le 1), \qquad M(t) = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+1}{\alpha+\beta+r}\right)\frac{t^k}{k!}$$

  Gamma(α, β)**:** $\mathbb{E}(X) = \alpha\beta$, $\mathbb{V}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

  $$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < 1/\beta$$

  Order Statistics: Let $X_{(1)} \le \cdots \le X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

  $$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

  $$f_{X_{(i)},X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1}[F(x_j) - F(x_i)]^{j-1-i}[1-F(x_j)]^{n-j} \times I(-\infty < x_i \le x_j < \infty)$$

  $$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1)\cdots f(x_n) I(-\infty < x_1 \le \cdots \le x_n < \infty)$$

1. Suppose $X \sim \text{Exp}(\lambda)$, where $\lambda > 0$, with pdf $f_X(x|\lambda) = \frac{1}{\lambda}e^{-x/\lambda}; x > 0$.
   (a) Find the pdf of $Y = \sqrt{X}$.
   (b) Find the mean and variance of Y.
   (c) Find the pdf of $Z = e^X$.
   (d) Give the conditions such that the first and second moments of Z exist.
   (e) Find the mean and variance of Z under the conditions obtained in (d).

2. Let X_1 and X_2 be a random sample from the density $f(x) = 1.5x^2; -1 < x < 1$.
   (a) Let $Y_1 = X_1$ and $Y_2 = X_1 + X_2$. Find the joint distribution of Y_1 and Y_2.
   (b) Find the correlation between Y_1 and Y_2.
   (c) Let $M = \max(X_1, X_2)$. Find the cdf of M.
   (d) Let $W = X_1X_2$. Find $\mathbb{E}(W)$ and $\mathbb{V}(W)$.

3. Let $U_1, U_2, \ldots$ be iid random variables having the uniform distribution on $[0, 1]$. Let $X_n = (\prod_{k=1}^n U_k)^{-1/n}$. Show that $\frac{\sqrt{n}(X_n - e)}{e} \xrightarrow{d} N(0, 1)$. (Note that "$\xrightarrow{d}$" refers to convergence in distribution, and e is Euler's constant.)

4. Suppose that $X_1, \ldots, X_n$ is an iid random sample from a $\text{Beta}(\alpha, 1)$ population.
   (a) Find a MOME for α.
   (b) Find the MLE for α.
   (c) Find the UMVUE for α.
   (d) Does the estimator found in (c) attain the Cramér-Rao Lower Bound? Justify your answer.

5. Suppose that $X_1, \ldots, X_n$ is an iid random sample from a $\text{Beta}(\theta, \theta)$ population. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where $\theta_1 > \theta_0$.
   (a) Give the form of the rejection region for the UMP level-α test. Simplify/reduce as much as possible.
   (b) Would this test also be UMP level-α for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$? Justify your answer.

6. A random variable X with cdf given by $F(x|\theta) = [1 - \exp(-x^2)]^\theta, x > 0, \theta > 0$ ($F(x|\theta) = 0$ if $x \leq 0$) is said to be a *Burr Type X* random variable ("BurrX(θ)").
   (a) Let $X \sim \text{BurrX}(\theta)$ and $Y \sim \text{BurrX}(\lambda)$. Show that $R = P(Y < X) = \frac{\theta}{\theta+\lambda} = \left(1 + \frac{\lambda}{\theta}\right)^{-1}$. (This quantity is known as the *Stress-Strength Reliability.*)
   (b) Prove that the distribution of $\ln\left(1 - e^{-X^2}\right)$ is exponential with mean $1/\theta$.
   (c) Let $X_1, \ldots, X_m$ be a random sample from a BurrX(θ) population, and $Y_1, \ldots, Y_n$ be a random sample from a BurrX(λ) population such that the samples are independent. Identify the distribution of $2\theta \sum_{i=1}^m -\ln\left(1 - e^{-X_i^2}\right)$. (The distribution of $2\lambda \sum_{j=1}^n -\ln\left(1 - e^{-Y_j^2}\right)$ would thus follow immediately.)
   (d) Prove that $\left(\frac{1}{R} - 1\right)\frac{\hat{\theta}}{\hat{\lambda}}$ is a pivotal quantity for R and identify its distribution, where $\hat{\theta} = \frac{m}{\sum_{i=1}^m -\ln\left(1-e^{-X_i^2}\right)}$ and $\hat{\lambda} = \frac{n}{\sum_{j=1}^n -\ln\left(1-e^{-Y_j^2}\right)}$.
   (e) Use the pivotal quantity in (d) to derive a $(1 - \alpha) \times 100\%$ confidence interval for R.