Probability and Statistics Preliminary Examination: May 2019

Instructions:

- Work all 6 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 6 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed); ind (independently distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I_{[x \in A]}$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbb{E}(X)$: expectation of random variable X.
  - $\mathbb{V}(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  Poisson(λ)**:** $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

  $$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  Geometric(p)**:** $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

  $$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$$

  Negative-Binomial(r, p)**:** $\mathbb{E}(X) = r(1-p)/p$, $\mathbb{V}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

  $$f(x) = \binom{r+x-1}{x} p^r (1-p)^x I(x \in \{0, 1, \cdots\}), \qquad M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

  Gamma(α, β)**:** $\mathbb{E}(X) = \alpha/\beta$, $\mathbb{V}(X) = \alpha/\beta^2$, and pdf and mgf given respectively by

  $$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1-t/\beta}\right)^\alpha, \quad t < \beta$$

  Multinomial$(n, \pi_1, \pi_2, ..., \pi_k)$**:** $\mathbb{E}(X_i) = n\pi_i$, $\mathbb{V}(X_i) = n\pi_i(1-\pi_i)$, and pdf is given by

  $$f(x_1, x_2, ..., x_k) = \frac{n!}{\prod_{j=1}^k x_j!} \prod_{j=1}^k \pi_k^{x_k}, \quad \sum_{j=1}^k \pi_j = 1; \quad \sum_{j=1}^k x_j = n$$

Dirichlet$(\alpha_1, \alpha_2, ..., \alpha_k)$**:** $\mathbb{E}(X_i) = \alpha_1 / \sum_{i=1}^k \alpha_i$, $\mathbb{V}(X_i) = \frac{\tilde{\alpha}_i(1-\tilde{\alpha}_i)}{\tilde{\alpha}+1}$, where $\tilde{\alpha} = \sum_{i=1}^k \alpha_i$ and $\tilde{\alpha}_i = \alpha_i/\tilde{\alpha}$ and pdf up to a normalizing constant is given by

$$f(x_1, x_2, ..., x_k) \propto \prod_{j=1}^k x_j^{\alpha_j - 1}, \quad \sum_{j=1}^k x_j = 1$$

Order Statistics: Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)},X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-1-i} [1 - F(x_j)]^{n-j} \times I(-\infty < x_i \leq x_j < \infty)$$

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \leq \cdots \leq x_n < \infty)$$

1. Flow of traffic at a certain traffic corner is modeled as a sequence of Bernoulli trials by assuming that the probability of a car passing during any given second is a constant p. Assume that the Bernoulli trials at each second is independent of any other trials. Suppose a pedestrian can cross the street only if no car is to pass during next 3 seconds. Let X a random variable denoting the waiting time of the pedestrian at that corner

   (a) Find $P(X \leq 1)$.

   (b) Find $P(X = 4)$.

2. Let $X_1, X_2 \overset{iid}{\sim}$ Uniform(0,1)

   (a) Find the pdf for $U = X_1 + X_2$.

   (b) Find the median of U.

   (c) Find $U_{0.25}$ and $U_{0.75}$.

3. Let $X_1, X_2, ..., X_k \overset{indep}{\sim}$ Poisson(λ_i), $\;i = 1, 2, ..., k$.

   (a) Find the conditional expectation, $E(X_i | \sum_{i=1}^{k} X_i = n)$

   (b) Find the conditional variance, $\mathbb{V}(X_i | \sum_{i=1}^{k} X_i = n)$

   (c) Find the conditional covariance, $Cov(X_i, X_j | \sum_{i=1}^{k} X_i = n)$

4. Let X be a single sample from Discrete Uniform (1,N), where $N \in \mathbb{N}$. We wish to estimate the unknown parameter N.

   (a) Show that $T(X) = X$ is complete.

   (b) Find the minimum variance unbiased estimator of N.

   (c) Now consider the parameter space to be $N \in \mathbb{N} - \{n\}$, i.e set of all positive integers except the interger n. Is $T(X)$ still complete? Why why not?

5. Let $X_1, X_2, ..., X_n \overset{iid}{\sim} Poisson(\lambda), \lambda > 0$.

   (a) Find the MLE of $\tau(\lambda) = P(X_i = 0)$.

   (b) Find the asymptotic distribution of $\tau(\lambda)$.

   (c) Obtain the asymptotic MSE of the MLE of $\tau(\lambda)$.

   (d) Now assume a Gamma(α, β) prior on λ. Assume α and β are known constants. Obtain the posterior mean of $\tau(\lambda)$.

6. Let $X_1, X_2, ...X_n \overset{iid}{\sim} Poisson(\lambda), \lambda > 0$. We wish to test $H_0 : \lambda = \lambda_0$ Vs. $H_a : \lambda > \lambda_0$.

   (a) Find the UMP level α test for the above pair of hypotheses.

   (b) Derive an approximate rejection region for the above hypothesis for large n.

   (c) Find the approximate power function, for large n, for this test.

   (d) Now suppose you wish to test $H_0 : \lambda = \lambda_0$ vs $H_a : \lambda \neq \lambda_0$. Derive an approximate test for this situation assuming large n. Is this test UMP, why, why not?