Probability and Statistics Preliminary Examination: August 2022

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I(x \in A)$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbb{E}(X)$: expectation of random variable X.
  - $\mathbb{V}(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  Unif(a, b)**:** $\mathbb{E}(X) = (a+b)/2$, $\mathbb{V}(X) = (b-a)^2/12$, and pdf given by

$$f(x) = (b-a)^{-1} I(a < x < b)$$

  Poisson(λ)**:** $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  Geometric(p)**:** $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

$$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -\log(1-p)$$

  Negative-Binomial(r, p)**:** $\mathbb{E}(X) = r(1-p)/p$, $\mathbb{V}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x I(x \in \{0, 1, \cdots\}), \qquad M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

  Gamma(α, β)**:** $\mathbb{E}(X) = \alpha\beta$, $\mathbb{V}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < 1/\beta$$

  Exp(β)**:** Gamma$(1, \beta)$.

  Beta(α, β)**:** $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathbb{V}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, and pdf given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} I(0 < x < 1)$$

Order Statistics: Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)},X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-1-i} [1 - F(x_j)]^{n-j} \times I(-\infty < x_i \leq x_j < \infty)$$

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \leq \cdots \leq x_n < \infty)$$

1. Let $X_1, \ldots, X_n$ be iid observations from Unif$(0, \theta^2)$ with $\theta > 0$.

   (a) Find the MOME of θ.

   (b) Let $\widehat{\theta}$ be the MOME in (a). Show that $\widehat{\theta} \xrightarrow{P} \theta$.

   (c) Find the limiting distribution of $\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{D} ?$

2. Suppose that $X_1, \ldots, X_n$ are iid random variables from Exp(β). However, we only observe the variable $Y_1, \ldots, Y_n$ where $Y_i = I(X_i > 1)$.

   (a) Base on the sample $Y_1, \ldots, Y_n$, find the MLE of β.

   (b) Let $\widehat{\beta}$ be the MLE in (a). Find the limiting distribution of $\sqrt{n}(\widehat{\beta} - \beta) \xrightarrow{D} ?$

3. Let $X_1, \ldots, X_n$ be iid observations with pdf $f(x) = 3x^2 I(0 < x < 1)$. Let $Y_n = min\{X_1, \ldots, X_n\}$.

   (a) Find the cdf of Y_n.

   (b) When $\alpha \in (0, 1/3)$, show that $n^{\alpha} Y_n \xrightarrow{P} 0$.

   (c) Find the limiting distribution of $n^{1/3} Y_n \xrightarrow{D} ?$

   (**Hint:** You may need the formula $\lim_{n\to\infty}(1 + a_n/n)^n = a$ if $\lim_{n\to\infty} a_n = a$.)

4. Let $X_1, \ldots, X_n$ be independent random variables from Exp(β). Let $T = \sum_{i=1}^{n} X_i$.

   (a) Are X_1/T and T independent? Why or why not.

   (b) Find the conditional pdf of X_1/T given $T = t$.

   (c) Find the UMVUE of β^2.

   (**Hint:** You may need the following result. Suppose $X \sim$Gamma(α_1, β) and $Y \sim$Gamma(α_2, β) are independent variables. Then $X/(X+Y) \sim$Beta(α_1, α_2).)

5. Let X_1, X_n be iid observations from Exp(θ). Consider the following two hypotheses, $H_0 : \theta = \theta_0$ V.S. $H_1 : \theta > \theta_0$.

   (a) Test 1: reject H_0 if $X_1 + X_2 > c_1$. Find the value of c_1 such that Test 1 has size α.

   (b) Test 2: reject H_0 if $\max\{X_1, X_2\} > c_2$. Find the value of c_2 such that Test 1 has size α.

   (c) Find the power function of Test 2.

   (d) Which test is more power? Briefly discuss the reason.