Probability and Statistics Preliminary Examination: August 2023

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I(x \in A)$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbb{E}(X)$: expectation of random variable X.
  - $\mathbb{V}(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  Bin(n,p)**:** $\mathbb{E}(X) = np$, $\mathbb{V}(X) = np(1-p)$, and pmf and mgf given respectively by

$$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} I(x \in \{0, 1, \cdots, n\}), \qquad M(t) = (1 - p + pe^t)^n$$

  Unif(a,b)**:** $\mathbb{E}(X) = (a+b)/2$, $\mathbb{V}(X) = (b-a)^2/12$, and pdf given by

$$f(x) = (b-a)^{-1} I(a < x < b)$$

  Poisson(λ)**:** $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  Geometric(p)**:** $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

$$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1 - (1-p)e^t}, \quad t < -\log(1-p)$$

  Negative-Binomial(r,p)**:** $\mathbb{E}(X) = r(1-p)/p$, $\mathbb{V}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x I(x \in \{0, 1, \cdots\}), \qquad M(t) = \left(\frac{p}{1-(1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

  Gamma(α,β)**:** $\mathbb{E}(X) = \alpha\beta$, $\mathbb{V}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < 1/\beta$$

Exp(β)**:** Gamma$(1, \beta)$.

Beta(α, β)**:** $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathbb{V}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, and pdf given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0 < x < 1)$$

Order Statistics: Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} [F(x)]^{j-1}[1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)}, X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1}[F(x_j) - F(x_i)]^{j-1-i}[1-F(x_j)]^{n-j} \times I(-\infty < x_i \leq x_j < \infty)$$

$$f_{X_{(1)}, \ldots, X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \leq \cdots \leq x_n < \infty)$$

1. Answer the following questions.

   (a) Let X and Y be two independent continuous random variables. Prove that

   $$P(X < Y) = \int_{-\infty}^{\infty} F_X(t) f_Y(t) dt.$$

   Here F_X is the cdf of X, and f_Y is the pdf of Y.

   (b) Find $P(X_3 > \min\{X_1, X_2\})$, where X_1, X_2, X_3 are i.i.d $Exp(\beta)$. (Note: $Exp(\beta)$ is the exponential distribution with mean β.)

2. Suppose $X_1, \cdots, X_n$ are i.i.d with uniform distribution $Unif(-\theta, \theta)$ with $\theta > 0$. Assume that the prior distribution of θ is $Unif(1, 2)$. Find the Bayesian estimator of θ.

3. Let $X_1, \ldots, X_n$ be i.i.d observations such that $\log(X_i) \sim N(\theta, \theta)$. Here log is the natural logarithm.

   (a) Find the MLE of θ.

   (b) Show that the MLE is consistent.

4. Let $X \sim Unif(a, b)$ with $0 < a < b < \infty$. Given $X = x$, the conditional distribution of $Y \sim Exp(x)$. Find the distribution of Y/X. (Note: $Exp(\beta)$ is the exponential distribution with mean β.)

5. Let $X_1, \ldots, X_n$ be i.i.d observations from $Unif(0, \theta)$. Consider the following two hypotheses, $H_0 : \theta = 1$ V.S. $H_1 : \theta > 1$.

   (a) Consider the test: reject H_0 if $\max\{X_1, \ldots, X_n\} > c$. Find the value of c such that the test has size α.

   (b) Find the power function of the test in (a).