Probability and Statistics Preliminary Examination: August 2024

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I(x \in A)$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbb{E}(X)$: expectation of random variable X.
  - $\mathbb{V}(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  Bin(n, p)**:** $\mathbb{E}(X) = np$, $\mathbb{V}(X) = np(1-p)$, and pmf and mgf given respectively by

  $$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} I(x \in \{0, 1, \cdots, n\}), \qquad M(t) = (1 - p + pe^t)^n$$

  Poisson(λ)**:** $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

  $$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  Geometric(p)**:** $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

  $$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1 - (1-p)e^t}, \quad t < -\log(1-p)$$

  Negative-Binomial(r, p)**:** $\mathbb{E}(X) = r(1-p)/p$, $\mathbb{V}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

  $$f(x) = \binom{r+x-1}{x} p^r (1-p)^x I(x \in \{0, 1, \cdots\}), \qquad M(t) = \left(\frac{p}{1 - (1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

  Gamma(α, β)**:** $\mathbb{E}(X) = \alpha\beta$, $\mathbb{V}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

  $$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1 - \beta t}\right)^\alpha, \quad t < 1/\beta$$

  Exp(β)**:** Gamma$(1, \beta)$.

Beta(α, β)**:** $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathbb{V}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, and pdf given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0 < x < 1)$$

Order Statistics: Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)}, X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1}[F(x_j) - F(x_i)]^{j-1-i}[1-F(x_j)]^{n-j} \times I(-\infty < x_i \leq x_j < \infty)$$

$$f_{X_{(1)}, \ldots, X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \leq \cdots \leq x_n < \infty)$$

1. Suppose that X has a pdf $f_X(x) = 1$ with $x \in (0,1)$, and Y has a pdf $f_Y(y) = e^{-y}$ with $y > 0$. Moreover, X and Y are independent. Let $W = \min(X, Y)$. Find $\mathbb{P}(W < t | W = X)$ for all $t \in \mathbb{R}$.

2. Let $X_1, \ldots, X_n$ be independent random variables with a common pmf $f(x|N) = 1/N$ with $x = 1, \ldots, N$. Here $N \in \{1, 2, \ldots\}$ is the parameter.

   (a) Find a minimal sufficient statistic for N.

   (b) Find the UMP test for $H_0 : N = N_0$ v.s. $H_1 : N > N_0$. Here $N_0 \geq 1$ is an known integer.

3. Let $X_1, \ldots, X_n$ be independent random variables from $Bin(1, \theta)$ with $\theta \in (0, 1)$. Let $\tau(\theta) = \theta(1-\theta)$ and $\overline{X}$ be the sample average.

   (a) Show that $W = \frac{n}{n-1}\overline{X}(1-\overline{X})$ is the UMVUE FOR $\tau(\theta)$.

   (b) When $\theta \neq 1/2$, find the limiting distribution of $\sqrt{n}(W - \tau(\theta)) \xrightarrow{D}$?

4. Let $X_1, \ldots, X_n$ be independent random variables from the common pdf $f(x|\theta) = \theta^3 x^2 e^{-\theta x}/2$ for $0 < x < \infty, 0 < \theta < \infty$. Let $\widehat{\theta} = 3n/\sum_{i=1}^{n} X_i$.

   (a) Find $\mathbb{E}(\widehat{\theta})$.

   (b) Find the limiting distribution $\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow{D} ?$

   (c) Based on the result in (b), find an asymptotic $1 - \alpha$ confidence interval for θ.

5. Let $X_1, \ldots, X_n$ be i.i.d observations from the pdf $f(x|\theta) = \frac{1}{2} I(\theta - 1 \leq x \leq \theta + 1)$.

   (a) Show that $X_{(n)} - \theta$ is a pivotal quantity.

   (b) Find a and b such that the interval $[X_{(n)} - a, X_{(n)} - b]$ is the shortest $1 - \alpha$ confidence interval.