Probability and Statistics Preliminary Examination: May 2025

Instructions:

- Work all 5 Problems. Neither calculators nor electronic devices of any kind are allowed. Clearly state any theorem or fact that you use. Each of the 5 Problems is equally weighted (but the parts within a Problem may not be).
- Abbreviations/Acronyms.
  - pmf (probability mass function); pdf (probability density function); cdf (cumulative distribution function); mgf (moment generating function); iid (independent and identically distributed).
  - MSE (mean squared error), MOME (method of moments estimator); MLE (maximum likelihood estimator); PBE (posterior Bayes estimator); UMVUE (uniform minimum variance unbiased estimator); UMP (uniformly most powerful); LRT (likelihood ratio test).
- Notation.
  - $I(x \in A)$ or $I_A(x)$: indicator function for set A; takes on the value 1 if $x \in A$ and 0 otherwise.
  - $\mathbb{E}(X)$: expectation of random variable X.
  - $\mathbb{V}(X)$: variance of random variable X.
  - $X \sim \mathrm{N}(a, b)$: X has a normal distribution with mean a and variance b.
- Common distributions and other results.

  Bin(n, p)**:** $\mathbb{E}(X) = np$, $\mathbb{V}(X) = np(1-p)$, and pmf and mgf given respectively by

$$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} I(x \in \{0, 1, \cdots, n\}), \qquad M(t) = (1 - p + pe^t)^n$$

  Poisson(λ)**:** $\mathbb{E}(X) = \lambda$, $\mathbb{V}(X) = \lambda$, and pmf and mgf given respectively by

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I(x \in \{0, 1, \cdots\}), \qquad M(t) = \exp\{\lambda(e^t - 1)\}$$

  Geometric(p)**:** $\mathbb{E}(X) = 1/p$, $\mathbb{V}(X) = (1-p)/p^2$, and pmf and mgf given respectively by

$$f(x) = p(1-p)^{x-1} I(x \in \{1, 2, \cdots\}), \qquad M(t) = \frac{pe^t}{1 - (1-p)e^t}, \quad t < -\log(1-p)$$

  Negative-Binomial(r, p)**:** $\mathbb{E}(X) = r(1-p)/p$, $\mathbb{V}(X) = r(1-p)/p^2$, and pmf and mgf, respectively:

$$f(x) = \binom{r + x - 1}{x} p^r (1-p)^x I(x \in \{0, 1, \cdots\}), \qquad M(t) = \left(\frac{p}{1 - (1-p)e^t}\right)^r, \quad t < -\log(1-p)$$

  Gamma(α, β)**:** $\mathbb{E}(X) = \alpha\beta$, $\mathbb{V}(X) = \alpha\beta^2$, and pdf and mgf given respectively by

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha - 1} e^{-x/\beta} I(x > 0), \qquad M(t) = \left(\frac{1}{1 - \beta t}\right)^\alpha, \quad t < 1/\beta$$

  Exp(β)**:** Gamma$(1, \beta)$.

Beta(α, β)**:** $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathbb{V}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, and pdf given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} I(0 < x < 1)$$

Order Statistics: Let $X_{(1)} \leq \cdots \leq X_{(n)}$ denote the order statistics from a random sample $X_1, \ldots, X_n$. If X_1 is continuous with pdf $f(x)$ and cdf $F(x)$, the pdf of $X_{(j)}$, $(X_{(i)}, X_{(j)})$, and $(X_{(1)}, \ldots, X_{(n)})$, are given by:

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j} f(x) I(-\infty < x < \infty)$$

$$f_{X_{(i)},X_{(j)}}(x_i, x_j) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(x_i) f(x_j) [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-1-i} [1 - F(x_j)]^{n-j} \times I(-\infty < x_i \leq x_j < \infty)$$

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1, \ldots, x_n) = n! f(x_1) \cdots f(x_n) I(-\infty < x_1 \leq \cdots \leq x_n < \infty)$$

1. Suppose that $X_1, \ldots, X_n$ are i.i.d $Bin(1, \theta)$ for $\theta \in \left[\frac{1}{4}, \frac{3}{4}\right]$. Find the MLE of θ.

   (**Remark:** Please show your work.)

2. Let X be a random variable with pdf

$$f(x|\theta) = \begin{cases} \frac{2x}{\theta} & \text{if } 0 \leq x \leq \theta, \\ \frac{2(1-x)}{1-\theta} & \text{if } \theta < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Here $\theta \in (0, 1)$.

   (a) Find u, l such that $\mathbf{P}(l \leq X \leq u|\theta) = 1 - \alpha$ and $u - l$ is minimized. Here $\alpha \in (0, 1/2)$ is a constant.

   (b) Based on the u, l calculated in (a), build a $1 - \alpha$ confidence interval for θ.

3. Suppose that $X_1, \ldots, X_n$ are i.i.d $Exp(\theta)$ with $\theta > 0$. Find the UMVUE of θ^{-2}.

   (**Remark:** $Exp(\theta)$ has a pdf $f(x) = \theta^{-1} e^{-x/\theta} I(x > 0)$.)

4. Let $X_1, \ldots, X_n$ be i.i.d $Unif(0, \theta)$ with $\theta > 0$. Suppose that conditioning on $X_1 = x_1, \ldots, X_n = x_n$, the random variable Y has a conditional distribution $Unif(0, 1/\overline{x})$, where $\overline{x} = \sum_{i=1}^n x_i/n$.

   (a) Let $V_n = \mathbf{Var}(Y|X_1, \ldots, X_n)$, the conditional variance of Y given $X_1, \ldots, X_n$. Find V_n.

   (b) Find a, b such that $\sqrt{n}(V_n - a) \xrightarrow{D} N(0, b)$.

5. Suppose that $X \sim Unif(0, \theta)$ with $\theta > 0$. Consider the hypotheses:

$$H_0 : 2 \leq \theta \leq 3, \quad V.S \quad H_1 : H_0 \text{ is false.}$$

   (a) Suppose we will reject H_0 if $X > 4$ or $X < 1$. Find size of the test.

   (**Remark:** The size of a test is defined as $\alpha = \sup_{\theta \in \Theta_0} \beta(\theta)$, where $\beta(\theta)$ is the power function.)

   (b) Now, suppose we will reject H_0 if $X > c$ or $X < 1$ for some constant $c > 1$. Find the value of c such that the test is of size $\alpha = \frac{2}{3}$.