**Real Analysis Preliminary Examination**

August, 2022

**DIRECTIONS:** Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded. If you do not do this, then problems 1-7 will be graded. Strive for clear and detailed solutions.

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**PROBLEMS:**

1. Let $\mu^*$ be an outer measure on $X$ and $E \subset X$. Prove that if for all $\epsilon > 0$, there is a $\mu^*$ measurable set $A \subset E$ such that $\mu^*(E \setminus A) < \epsilon$, then $E$ is $\mu^*$-measurable.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f, g$ be real valued measurable functions on $X$ such that $g(x) \neq 0 \ \forall x \in X$. Prove that $f/g$ is measurable.

3. Let $\mathcal{A}$ be a semi-algebra over $X$, $\mu$ be a pre-measure on $\mathcal{A}$, and $\mu^*$ be the outer measure on $X$ induced by $\mu$.

   a) Prove that for every $E \subset X$, there is a $\mu^*$-measurable $A \supset E$ such that
   $$\mu^*(E) = \mu^*(A).$$

   b) Prove that $B \subset X$ is $\mu^*$-measurable if and only if for every $\mu^*$-measurable set $A$ with $\mu^*(A) < \infty$,
   $$\mu^*(A) = \mu^*(A \cup B) + \mu^*(A \cup B^c).$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f(x)$ be a real valued measurable function on $X$ such that
   $$\int_X |f| \, d\mu = 0.$$
   Prove that $f = 0$ a.e.

5. Compute (with justification) $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{\cos x}{x^{1/n} + x^n} \, dx.$

6. Let $S \subset \mathbb{R}^2$ be the region defined by
   $$S = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq \sin(x), 0 < x < \pi\}.$$
   For any $f \in L^1((0, \pi), m)$, prove that $\csc(x) f(x) \in L^1(S, m \times m)$ and
   $$\int_S \csc(x) f(x) \, d(m \times m) = \int_0^\pi f(x) \, dx.$$

7. Prove in an infinite dimensional Hilbert space $H$, the unit ball

$$B = \{h \in H : \|h\| \leq 1\}$$

is not compact.

8. Let $X$ and $Y$ be Banach spaces over $\mathbb{R}$. Prove that if a bounded linear operator $T : X \to Y$ is one-to-one and onto, then $T^{-1} : Y \to X$ is a bounded LINEAR operator.

9. Let $0 < p, q, r < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Prove that for any measurable functions $f, g$ over $(X, \mathcal{M}, \mu)$,

$$\|fg\|_r \leq \|f\|_p \|g\|_q.$$